Optimization Methods in Finance - Homework I (Solutions)

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1 Problems

1. **Convexity:**

   (a) Let S denote a set and let $\mathbf{x}$ denote a point of S. Argue that $\mathbf{x}$ is an extreme point of S, if and only if $S - \{\mathbf{x}\}$ is convex.

   (b) Consider the linear program:

$$\begin{array}{ccc} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

   Let $\mathbf{x_1}$ and $\mathbf{x_2}$ represent two optimal solutions for the above linear program. Argue that the parametric point $(\alpha \cdot \mathbf{x_1} + (1-\alpha) \cdot \mathbf{x_2})$, $\alpha \in [0,1]$ is also an optimal solution for the linear program.

   (c) Given two convex sets S_1 and S_2, what can you say about the sets $S_1 \cup S_2$ and $S_1 \cap S_2$ as regards convexity.

   Solution:

   (a) Let S be a convex set, and let $\mathbf{x}$ be an extreme point of S. Consider the set $S - \{\mathbf{x}\}$. Let $\mathbf{z_1}$ and $\mathbf{z_2}$ be any two points in $S - \{\mathbf{x}\}$, and let $\alpha \in [0,1]$. We show that $(\alpha \cdot \mathbf{z_1} + (1-\alpha) \cdot \mathbf{z_2}) \in (S - \{\mathbf{x}\})$. Suppose not. Since S is a convex set, we have that $(\alpha \cdot \mathbf{z_1} + (1-\alpha) \cdot \mathbf{z_2}) \in S$, hence we have that $\mathbf{x} = \alpha \cdot \mathbf{z_1} + (1-\alpha) \cdot \mathbf{z_2}$. Since $\mathbf{z_1} \neq \mathbf{x}$ and $\mathbf{z_2} \neq \mathbf{x}$, we get a contradiction to $\mathbf{x}$ being an extreme point of S.

   Conversely, assume that the set $S - \{\mathbf{x}\}$ is convex. We claim that $\mathbf{x}$ is an extreme point of S. Suppose not. Then there are $\mathbf{z_1}, \mathbf{z_2} \in S$, $\mathbf{z_1} \neq \mathbf{x}$ and $\mathbf{z_2} \neq \mathbf{x}$ such that $\mathbf{x} = \alpha \cdot \mathbf{z_1} + (1-\alpha) \cdot \mathbf{z_2}$ for some $\alpha \in (0,1)$. Clearly, $\mathbf{z_1}, \mathbf{z_2} \in S - \{\mathbf{x}\}$ and $\mathbf{x} \notin S - \{\mathbf{x}\}$. Hence the set $S - \{\mathbf{x}\}$ is not convex contradicting our assumption.

   (b) Let c^* be the optimum value of the linear program in hand. We have that

$$\mathbf{c} \cdot \mathbf{x_1} = \mathbf{c} \cdot \mathbf{x_2} = c^*.$$

   Consider the parametric point $(\alpha \cdot \mathbf{x_1} + (1-\alpha) \cdot \mathbf{x_2})$, $\alpha \in [0,1]$. We have that

$$\begin{aligned} \mathbf{c} \cdot (\alpha \cdot \mathbf{x_1} + (1-\alpha) \cdot \mathbf{x_2}) &= \alpha \cdot \mathbf{c} \cdot \mathbf{x_1} + (1-\alpha) \cdot \mathbf{c} \cdot \mathbf{x_2} \\ &= \alpha \cdot c^* + (1-\alpha) \cdot c^* \\ &= c^*. \end{aligned}$$

   Moreover, since $\alpha \in [0,1]$, we also have that

$$\begin{aligned}\mathbf{A}\cdot(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2}) &= \alpha\cdot\mathbf{A}\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{A}\cdot\mathbf{x_2}\\ &\le \alpha\cdot\mathbf{b}+(1-\alpha)\cdot\mathbf{b}\\ &= \mathbf{b},\end{aligned}$$

and

$$(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})\ge 0.$$

This implies that the parametric point $(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})$, $\alpha\in[0,1]$ is also an optimal solution for the linear program.

(c) The union of two convex sets is not necessarily convex. Take two disjoint convex sets, and take one point from each of them. Then, the line connecting these points has a point that lies outside the union of the sets.

On the positive side, the intersection of two convex sets is always convex. Let S_1 and S_2 be two convex sets, and let $\mathbf{x_1},\mathbf{x_2}\in S_1\cap S_2$. Consider the parametric point $(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})$, $\alpha\in[0,1]$. Clearly, $\mathbf{x_1},\mathbf{x_2}\in S_1$ and $\mathbf{x_1},\mathbf{x_2}\in S_2$, and since both of the sets are convex, we also have that $(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})\in S_1$ and $(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})\in S_2$. Thus, $(\alpha\cdot\mathbf{x_1}+(1-\alpha)\cdot\mathbf{x_2})\in S_1\cap S_2$.

□

2. **Linear Programming:**

Consider the following linear program:

$$\begin{aligned}\max z = x_1+2\cdot x_2-9\cdot x_3+8\cdot x_4-36\cdot x_5 & \\ 2\cdot x_2-x_3+x_4-3\cdot x_5 &\le 40\\ x_1-x_2+2\cdot x_4-2\cdot x_5 &\le 10\\ x_1,x_2,x_3,x_4,x_5 &\ge 0\end{aligned}$$

(a) Solve the problem using the Simplex algorithm, showing all the steps.

(b) Write down the dual of the above problem.

(c) Solve the dual graphically and then solve the primal using complementary slackness.

Solution:

(a) We add slack variables s_1 and s_2 to the program, so that we have it in canonical form.

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	
Z	-1	-2	9	-8	36	0	0	0
S_1	0	2	-1	1	-3	1	0	40
S_2	1	-1	0	2	-2	0	1	10

The next tableau is the following:

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	
Z	3	-6	9	0	28	0	4	40
S_1	-0.5	2.5	-1	0	4	1	-0.5	35
X_4	0.5	-0.5	0	1	-1	0	0.5	5

After one iteration, we get:

	x_1	x_2	x_3	x_4	x_5	s_1	s_2	
Z	1.8	0	6.6	0	37.6	2.4	2.8	124
X_2	-0.2	1	-0.4	0	1.6	0.4	-0.2	14
X_4	0.4	0	0.2	1	-0.2	0.2	0.4	12

We have reached the final point. The optimal value is 124 and $X=[0,14,0,12,0]$.

(b) The dual program of the above mentioned linear program is given below:

$$\begin{aligned} \min z = 40 \cdot y_1 + 10 \cdot y_2 & \\ y_2 &\geq 1 \\ 2 \cdot y_1 - y_2 &\geq 2 \\ -y_1 &\geq -9 \\ y_1 + 2 \cdot y_2 &\geq 8 \\ -3 \cdot y_1 - 2 \cdot y_2 &\geq -36 \\ y_1, y_2 &\geq 0 \end{aligned}$$

(c) The graphical solution to the dual, is shown in Figure 1.

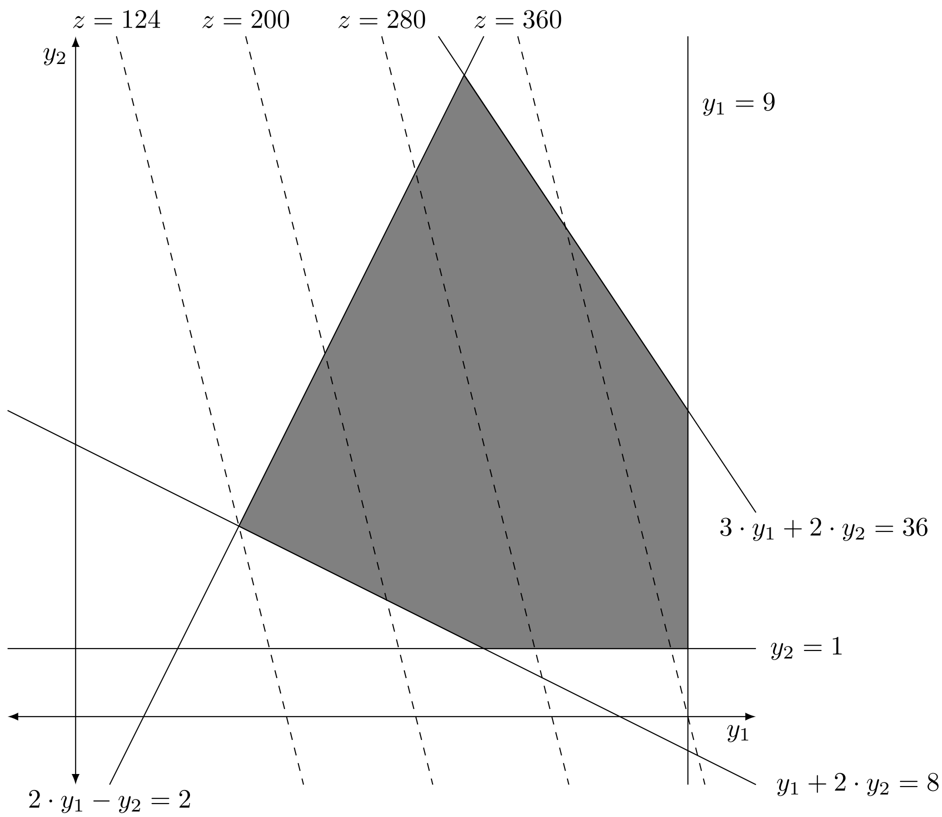

Figure 1: Graphical solution to the dual program

From this figure we get that $(y_1, y_2) = (2.4, 2.8)$. Moreover, the constraints $2 \cdot y_1 - y_2 \geq 2$ and $y_1 + 2 \cdot y_2 \geq 8$ are satisfied with equality. Thus, both primal constraints are met with equality and only x_2 and x_4 can be non-zero. Thus, the primal optimum satisfies $2 \cdot x_2 + x_4 = 40$ and $-x_2 + 2x_4 = 10$. This gives us a solution of $(x_1, x_2, x_3, x_4, x_5) = (0, 14, 0, 12, 0)$.

□

3. **Non-linear Programming (Theory):**

   (a) Solve the following NLP analitically:

$$\min(x_2 + x_3 - 5)^2 - \frac{(x_3 + x_4 + 2)}{7} + 4$$

$$-x_1 + x_2 + x_3 - 5 = 0$$
$$-3 \cdot x_1 - x_2 + x_3 - 3 = 0$$
$$-7 \cdot x_1 + x_3 + x_4 + 2 = 0$$
$$x_1 \leq 1, x_2 \leq 1, x_3 \leq 6, x_4 \leq 1$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

(b) Check the point $(1, 2)$ for first order necessary conditions (KKT conditions):

$$\max 2 \cdot x_1^3 + 3 \cdot x_2^4$$
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 3$$
$$x_2 - x_1 \leq 1$$
$$x_1 - x_2 \geq -1$$
$$x_1 \geq 0, x_2 \geq 0.$$

Solution:

(a) By adding the first two constraints, and solving with respect to x_1, we get

$$x_3 = 2 \cdot x_1 + 4.$$

Plugging this into the third constraint, and solving with respect to x_1, we get:

$$x_4 = 5 \cdot x_1 - 6.$$

Now, taking into account that $0 \leq x_1 \leq 1$, we have that $-6 \leq x_4 \leq -1$. Thus the program has no feasible solutions, because we have that $0 \leq x_4 \leq 1$.

(b) We first write the program as a minimization problem, and change all constraints to the form ≥ 0. Taking into account that there are two constraints that are equivalent, we get:

$$\min(-2 \cdot x_1^3 - 3 \cdot x_2^4)$$
$$x_1 + x_2 - 1 \geq 0$$
$$-x_1 - x_2 + 3 \geq 0$$
$$x_1 - x_2 + 1 \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

The Lagrangian of the resulting program is:

$$L(x_1, x_2, \lambda) = -2 \cdot x_1^3 - 3 \cdot x_2^4 - \lambda_1 \cdot (x_1 + x_2 - 1) - \lambda_2 \cdot (-x_1 - x_2 + 3) - \lambda_3 \cdot (x_1 - x_2 + 1) - \lambda_4 \cdot x_1 - \lambda_5 \cdot x_2.$$

The first order KKT conditions for this program are:

$$\frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) = -6 \cdot x_1^2 - \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$
$$\frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) = -12 \cdot x_2^3 - \lambda_1 + \lambda_2 + \lambda_3 - \lambda_5 = 0$$
$$\lambda_1 \cdot (x_1 + x_2 - 1) = 0$$

$$
\begin{aligned}
\lambda_2 \cdot (-x_1 - x_2 + 3) = 0 \\
\lambda_3 \cdot (x_1 - x_2 + 1) = 0 \\
\lambda_4 \cdot x_1 = 0 \\
\lambda_5 \cdot x_2 = 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
\end{aligned}
$$

We need to check the point $(1, 2)$ for the first order KKT conditions, so we plug the values of x_1 and x_2 in the program. We get:

$$
\begin{aligned}
-6 - \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \\
-96 - \lambda_1 + \lambda_2 + \lambda_3 - \lambda_5 = 0 \\
\lambda_1 \cdot 2 = 0 \\
\lambda_2 \cdot 0 = 0 \\
\lambda_3 \cdot 0 = 0 \\
\lambda_4 \cdot 1 = 0 \\
\lambda_5 \cdot 2 = 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
\end{aligned}
$$

These conditions imply that $\lambda_1 = 0$, $\lambda_4 = 0$, $\lambda_5 = 0$. By plugging these values into the first two constraints, we get:

$$
\begin{aligned}
-6 + \lambda_2 - \lambda_3 = 0 \\
-96 + \lambda_2 + \lambda_3 = 0 \\
\lambda_2, \lambda_3, \geq 0
\end{aligned}
$$

By solving the system of linear equations, we get $\lambda_2 = 51$ and $\lambda_3 = 45$. We see that the values found for λs are non-negative, hence the point $(1, 2)$ meets the first order KKT conditions.

□

4. **Non-linear Programming (Applications):**

The partial derivative $\frac{\partial f(x)}{\partial x_i}$ of the function $f(x)$ with respect to the ith coordinate of the x vector can be estimated as

$$\frac{\partial f(x)}{\partial x_i} \approx \frac{f(x + h \cdot e_i) - f(x)}{h},$$

where e_i denotes the ith unit vector. Assuming that f is continuously differentiable, provide an upper bound on the estimation error from this finite-difference approximation using a Taylor series expansion for the function f around x. Next compute a similar bound for the alternative finite-difference formula given by

$$\frac{\partial f(x)}{\partial x_i} \approx \frac{f(x + h \cdot e_i) - f(x - h \cdot e_i)}{2 \cdot h}.$$

Comment on potential advantages and disadvantages of these two approaches.

Solution: Using the Taylor series expansion for the function f around x, we have the following equality:

$$f(x + h \cdot e_i) = f(x) + \frac{\partial f(x)}{\partial x_i} \cdot h + \frac{1}{2!} \cdot \frac{\partial^2 f(x)}{\partial x_i^2} \cdot h^2 + O(h^3).$$

By replacing h with $-h$ in this expression, we get:

$$f(x - h \cdot e_i) = f(x) - \frac{\partial f(x)}{\partial x_i} \cdot h + \frac{1}{2!} \cdot \frac{\partial^2 f(x)}{\partial x_i^2} \cdot h^2 + O(h^3).$$

When $h \to 0$, we have that

$$\frac{f(x + h \cdot e_i) - f(x)}{h} = \frac{\partial f(x)}{\partial x_i} + O(h),$$

and

$$\frac{f(x + h \cdot e_i) - f(x - h \cdot e_i)}{2 \cdot h} = \frac{\partial f(x)}{\partial x_i} + O(h^2).$$

The last two equations imply that the second finite-difference formula is more preferable than the first one, since it has $O(h^2)$ error-term in contrast with the first one which has that of $O(h)$.

□

5. **Quadratic Programming:**

   (a) Consider the quadratic function $f(x) = c^T \cdot x + \frac{1}{2} \cdot x^T \cdot Q \cdot x$, where the matrix Q is $n \times n$ and symmetric.

   i. Prove that if $x^T \cdot Q \cdot x < 0$ for some x, then f is unbounded below.

   ii. Prove that if Q is positive semidefinite (but not positive definite), then either f is unbounded below or it has an infinite number of solutions.

   iii. True or False: f has a unique minimizer if and only if Q is positive definite.

   (b) Consider the following quadratic program:

$$\begin{array}{ll} \min & x_1 \cdot x_2 + x_1^2 + \frac{3}{2} x_2{}^2 + 2 \cdot x_3^2 \\ & +2 \cdot x_1 + x_2 + 3 \cdot x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1 - x_2 = 0 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

   i. Is the optimization function convex?

   ii. Is the point $(\frac{1}{2}, \frac{1}{2}, 0)$ optimal? Provide a rigorous argument for your answer.

Solution:

(a) i. Assume that there is x, so that $x^T \cdot Q \cdot x < 0$. Since $0^T \cdot Q \cdot 0 = 0$, we have that $x \neq 0$. Consider arbitrary number $\alpha \neq 0$. We have that

$$\begin{aligned} f(\alpha \cdot x) &= c^T \cdot (\alpha \cdot x) + \frac{1}{2} \cdot (\alpha \cdot x)^T \cdot Q \cdot (\alpha \cdot x) \\ &= \alpha \cdot (c^T \cdot x) + \frac{1}{2} \cdot \alpha^2 \cdot (x^T \cdot Q \cdot x). \end{aligned}$$

Since $x^T \cdot Q \cdot x < 0$, we get that $f(\alpha \cdot x) \to -\infty$ when $\alpha \to \infty$.

ii. Assume that Q is positive semidefinite, but not positive definite. Then there is $x \neq 0$, such that $x^T \cdot Q \cdot x = 0$. Consider arbitrary number $\alpha \neq 0$, and the vector $\alpha \cdot x$. We have that $\alpha \cdot x \neq 0$. Now, by the equality proved above, we have that

$$\begin{aligned} f(\alpha \cdot x) &= \alpha \cdot (c^T \cdot x) + \frac{1}{2} \cdot \alpha^2 \cdot (x^T \cdot Q \cdot x) \\ &= \alpha \cdot (c^T \cdot x). \end{aligned}$$

We consider three cases. If $(c^T \cdot x) > 0$, then clearly $f(\alpha \cdot x) \to -\infty$ when $\alpha \to -\infty$. Hence f is unbounded in this case. If $(c^T \cdot x) < 0$, then clearly $f(\alpha \cdot x) \to -\infty$ when $\alpha \to +\infty$. Hence f is unbounded in this case as well. Finally, if $(c^T \cdot x) = 0$, then clearly $f(\alpha \cdot x) = 0$, for any α, hence we have that f has infinitely many zeros.

iii. We claim that the answer to this statement is True. That is, we claim that f has a unique minimizer if and only if Q is positive definite.
One direction is easy to prove. Assume that Q is positive definite. Then $|Q| > 0$, hence $|Q| \neq 0$. We have that $\bigtriangledown f = c^T + Q \cdot x$ and $\bigtriangledown^2 f = Q$. Since $|Q| \neq 0$, we have that $\bigtriangledown f = 0$ has a unique solution, and as $\bigtriangledown^2 f = Q$ is positive definite, we have that this solution is a unique minimizer of the function f.
For the proof of the converse statement, assume that x^* is the unique minimizer of f. By the first part of this assignment, we have that Q is positive semi-definite. We claim that Q is positive definite. Assume that there is a vector $l \neq 0$, such that $l^T \cdot Q \cdot l = 0$. By the second part of this assignment, we have that $c^T \cdot l = 0$. Consider the vector $(x^* + l)$. We have that:

$$\bigtriangledown f(x^*) = c^T + Q \cdot x^* = 0,$$

and

$$\begin{aligned} f(x^* + l) &= c^T \cdot (x^* + l) + \frac{1}{2} \cdot (x^* + l)^T \cdot Q \cdot (x^* + l) \\ &= c^T \cdot x^* + c^T \cdot l + \frac{1}{2} \cdot [(x^*)^T \cdot Q \cdot (x^*) + 2 \cdot l^T \cdot Q \cdot x^* + l^T \cdot Q \cdot l] \\ &= f(x^*) + c^T \cdot l + l^T \cdot Q \cdot x^* + \frac{1}{2} \cdot l^T \cdot Q \cdot l \\ &= f(x^*) + c^T \cdot l - l^T \cdot c + \frac{1}{2} \cdot l^T \cdot Q \cdot l \\ &= f(x^*) \end{aligned}$$

Since $l \neq 0$, we have that x^* is not a unique minimizer of f, contradicting our assumption.

(b) i. The Hessian of the optimization function f is

$$\frac{\partial^2 f}{\partial x^2} = \left\| \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{array} \right\|$$

Since

$$\left| \begin{array}{c} 2 \end{array} \right| = 2 > 0,$$

$$\left| \begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array} \right| = 5 > 0,$$

$$\left| \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{array} \right| = 20 > 0,$$

we have that the Hessian of the optimization function f is positive definite, hence f is a convex function.

ii. First, we check for feasibility. We have that

$$\begin{aligned} \frac{1}{2} + \frac{1}{2} + 0 &= 1 \\ \frac{1}{2} - \frac{1}{2} &= 0 \\ \frac{1}{2}, \frac{1}{2}, 0 &\geq 0 \end{aligned}$$

Thus primal conditions are satisfied.
Now we need to check the remaining optimality conditions. There should exist $\mathbf{y}$ and $\mathbf{s}$ such that:

$$
\begin{aligned}
y_1 + y_2 - 2 \cdot x_1 - x_2 + s_1 = 2 \\
y_1 - y_2 - x_1 - 3 \cdot x_2 + s_2 = 1 \\
y_1 - 4 \cdot x_3 + s_3 = 3 \\
s_1, s_2, s_3 \geq 0 \\
s_1 \cdot x_1 = 0,\ s_2 \cdot x_2 = 0,\ s_3 \cdot x_3 = 0
\end{aligned}
$$

Taking into account that $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$, we get:

$$
\begin{aligned}
y_1 + y_2 + s_1 = \frac{7}{2} \\
y_1 - y_2 + s_2 = 3 \\
y_1 + s_3 = 3 \\
s_1, s_2, s_3 \geq 0 \\
s_1 = 0,\ s_2 = 0,\ 0 = 0
\end{aligned}
$$

These conditions imply that $(y_1, y_2) = \left(\frac{13}{4}, \frac{1}{4}\right)$ and $s_3 = 3 - y_1 = -\frac{1}{4} < 0$. Thus, the optimality conditions are not satisfied. This means that $(\frac{1}{2}, \frac{1}{2}, 0)$ is not an optimal solution.

□