Optimization Methods in Finance - Homework II

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1 Instructions

1. The homework is due on April 20.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. Consider the cone defined as:

$$C_q^r = \{(x_1, x_2, \dots x_n) : 2 \cdot x_1 \cdot x_2 \geq \sum_{j=3}^{n} x_j^2,\ x_1, x_2 \geq 0.\}$$

   Show that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in C_q^r$ if and only if $\mathbf{y} = (y_1, y_2, \dots, y_n) \in C_q$, where, $y_1 = (\frac{1}{\sqrt{2}}) \cdot (x_1 + x_2)$, $y_2 = (\frac{1}{\sqrt{2}}) \cdot (x_1 - x_2)$, $y_j = x_j$, $j = 3, 4, \dots n$ and C_q is the Lorenz cone given by:

$$C_q = \{\mathbf{y} = (y_1, y_2, \dots, y_n) \in \Re^n : y_1 \geq ||(y_2, y_3, \dots, y_n)||_2\}.$$

2. Assume that $-u_1 \leq f_1(\mathbf{x}) \leq u_1$ and $-u_2 \leq f_2(\mathbf{x}) \leq u_2$. Use integer programming to model the following conditions:
   (a) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$.
   (b) $f_1(\mathbf{x}) \geq 0 \rightarrow f_2(\mathbf{x}) \geq 0$.
   (c) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$, but not both.
   (d) $|\sum_{i=1}^{n} a_i \cdot x_i| \geq b$, where $b > 0$.

3. Consider the following integer programming problem:

$$\max z = 2 \cdot x_1 + x_2$$

subject to

$$2 \cdot x_1 - 2 \cdot x_2 \leq 3$$
$$-2 \cdot x_1 + x_2 \leq 2$$
$$2 \cdot x_1 + 2 \cdot x_2 \leq 13$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \text{ integer}$$

(a) Solve the above problem graphically.

(b) Solve the above problem using the branch-and-bound technique discussed in class. All linear programming relaxations should be solved graphically.

4. The following problem is called the stagewise shortest path problem:
You are given n cities, which are partitioned into $(N + 1)$ stages. City O is the only city in Stage 0 and city D is the only city in Stage N. Each city in stage k can advance to any city in stage $k + 1$, for $k = 0, 1, 2, \dots (N - 1)$. The distance between city i and city j is denoted by d_{ij}.

   Formulate the dynamic programming recursion for the stagewise shortest path problem which is the problem of finding the shortest path from the city O to the city D.

5. Compute the value of an American put option on a stock with current price equal to $\$100$, strike price equal to $\$98$, and expiration date five weeks from today. The yearly volatility of the logarithm of the stock return is $\sigma = 0.30$. The risk-free interest rate is 4%. Use a binomial lattice with $N = 5$.