Optimization Methods in Finance - Homework II (Solutions)

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Problems

1. Consider the cone defined as:

$$C_q^r = \{(x_1, x_2, \ldots x_n) : 2 \cdot x_1 \cdot x_2 \geq \sum_{j=3}^{n} x_j^2,\ x_1, x_2 \geq 0.\}$$

Show that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C_q^r$ if and only if $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in C_q$, where, $y_1 = (\frac{1}{\sqrt{2}}) \cdot (x_1 + x_2)$, $y_2 = (\frac{1}{\sqrt{2}}) \cdot (x_1 - x_2)$, $y_j = x_j$, $j = 3, 4, \ldots n$ and C_q is the Lorenz cone given by:

$$C_q = \{\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Re^n : y_1 \geq ||(y_2, y_3, \ldots, y_n)||_2\}.$$

Solution:
Assume that $\mathbf{x} \in C_q^r$. Since $y_1 = \frac{x_1+x_2}{\sqrt{2}}$ and $y_2 = \frac{x_1-x_2}{\sqrt{2}}$, we have that $x_1 = \frac{y_1+y_2}{\sqrt{2}}$ and $x_2 = \frac{y_1-y_2}{\sqrt{2}}$. Thus, we get:

$$\begin{aligned} 2 \cdot x_1 \cdot x_2 &\geq \sum_{j=3}^{n} x_j^2 \\ 2 \cdot (\frac{y_1 + y_2}{\sqrt{2}}) \cdot (\frac{y_1 - y_2}{\sqrt{2}}) &\geq \sum_{j=3}^{n} y_j^2 \\ y_1^2 - y_2^2 &\geq \sum_{j=3}^{n} y_j^2 \\ y_1^2 &\geq \sum_{j=2}^{n} y_j^2 \end{aligned}$$

As $x_1, x_2 \geq 0$ we have that $y_1 \geq 0$. Thus, we get:

$$y_1 \geq \sqrt{\sum_{j=2}^{n} y_j^2} = ||(y_2, y_3, \ldots, y_n)||_2$$

This means that, if $\mathbf{x} \in C_q^r$, then $\mathbf{y} \in C_q$.
Now, assume that $\mathbf{y} \in C_q$. From

$$y_1 \geq ||(y_2, y_3, \ldots, y_n)||_2$$

we have $y_1 \geq 0$ and $y_1^2 \geq \sum_{j=2}^n y_j^2$. Thus, $y_1^2 - y_2^2 \geq \sum_{j=3}^n y_j^2$ which means that $y_1^2 - y_2^2 \geq 0$. Since $y_1 \geq 0$, we have that $y_1 \geq |y_2|$. Thus $x_1, x_2 \geq 0$. We also have that

$$\begin{aligned}
y_1^2 - y_2^2 &\geq \sum_{j=3}^n y_j^2 \\
(\frac{x_1 + x_2}{\sqrt{2}})^2 - (\frac{x_1 - x_2}{\sqrt{2}})^2 &\geq \sum_{j=3}^n x_j^2 \\
\frac{x_1^2 + 2 \cdot x_1 \cdot x_2 + x_2^2}{2} - \frac{x_1^2 - 2 \cdot x_1 \cdot x_2 + x_2^2}{2} &\geq \sum_{j=3}^n x_j^2 \\
2 \cdot x_1 \cdot x_2 &\geq \sum_{j=3}^n x_j^2
\end{aligned}$$

This means that, if $\mathbf{y} \in C_q$, then $\mathbf{x} \in C_q^r$.

□

2. Assume that $-u_1 \leq f_1(\mathbf{x}) \leq u_1$ and $-u_2 \leq f_2(\mathbf{x}) \leq u_2$. Use integer programming to model the following conditions:

   (a) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$.

   (b) $f_1(\mathbf{x}) \geq 0 \rightarrow f_2(\mathbf{x}) \geq 0$.

   (c) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$, but not both.

   (d) $|\sum_{i=1}^n a_i \cdot x_i| \geq b$, where $b > 0$.

   Solution:

   y_1 and y_2 are defined here and they will be used in the other following.

$$y_1 = \begin{cases} 1, & \text{if } f_1(x) \geq 0 \\ 0, & \text{if } f_1(x) < 0. \end{cases} \quad y_2 = \begin{cases} 1, & \text{if } f_2(x) \geq 0 \\ 0, & \text{if } f_2(x) < 0. \end{cases} \tag{1}$$

   (a) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$.
   Solution: so we can have $\{0, 1\}$ and $\{1, 0\}$ and $\{1, 1\}$ but not $\{0, 0\}$, so the corresponding integer program is going to be:

$$\begin{aligned}
0 \leq y_1 \leq 1 \\
0 \leq y_2 \leq 1 \\
y_1 + y_2 \geq 1 \\
y_1, y_2 \in Z.
\end{aligned}$$

   (b) $f_1(\mathbf{x}) \geq 0 \rightarrow f_2(\mathbf{x}) \geq 0$.
   Solution: So we have $\{1, 1\}$ and $\{0, 0\}$ and $\{1, 1\}$ but not $\{1, 0\}$ so the corresponding integer program is going to be:

$$\begin{aligned}
0 \leq y_1 \leq 1 \\
0 \leq y_2 \leq 1 \\
y_1 - y_2 \leq 0 \\
y_1, y_2 \in Z.
\end{aligned}$$

(c) Either $f_1(\mathbf{x}) \geq 0$ or $f_2(\mathbf{x}) \geq 0$, but not both.
Solution: So we can have $\{0,1\}$ and $\{1,0\}$ but not $\{1,1\}$ nor $\{0,0\}$, so the corresponding integer program is going to be:

$$\begin{aligned} 0 \leq y_1 &\leq 1 \\ 0 \leq y_2 &\leq 1 \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\in Z. \end{aligned}$$

(d) $|\sum_{i=1}^{n} a_i \cdot x_i| \geq b$, where $b > 0$.
We know that absolute value could be formed like this:

$$y_1 = \begin{cases} 1, & \text{if } \sum(a_i x_i) \geq b. \\ 0, & \text{otherwise.} \end{cases} \quad y_2 = \begin{cases} 1, & \text{if } \sum(a_i x_i) \leq -b. \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Here we can have $\{1,0\}$ and $\{0,1\}$ but we cannot have $\{0,0\}$ and also it is not possible to have $\{1,1\}$, so the corresponding integer program is going to be:

$$\begin{aligned} 0 \leq y_1 &\leq 1 \\ 0 \leq y_2 &\leq 1 \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\in Z. \end{aligned}$$

□

3. Consider the following integer programming problem:

$$\max z = 2 \cdot x_1 + x_2$$

subject to

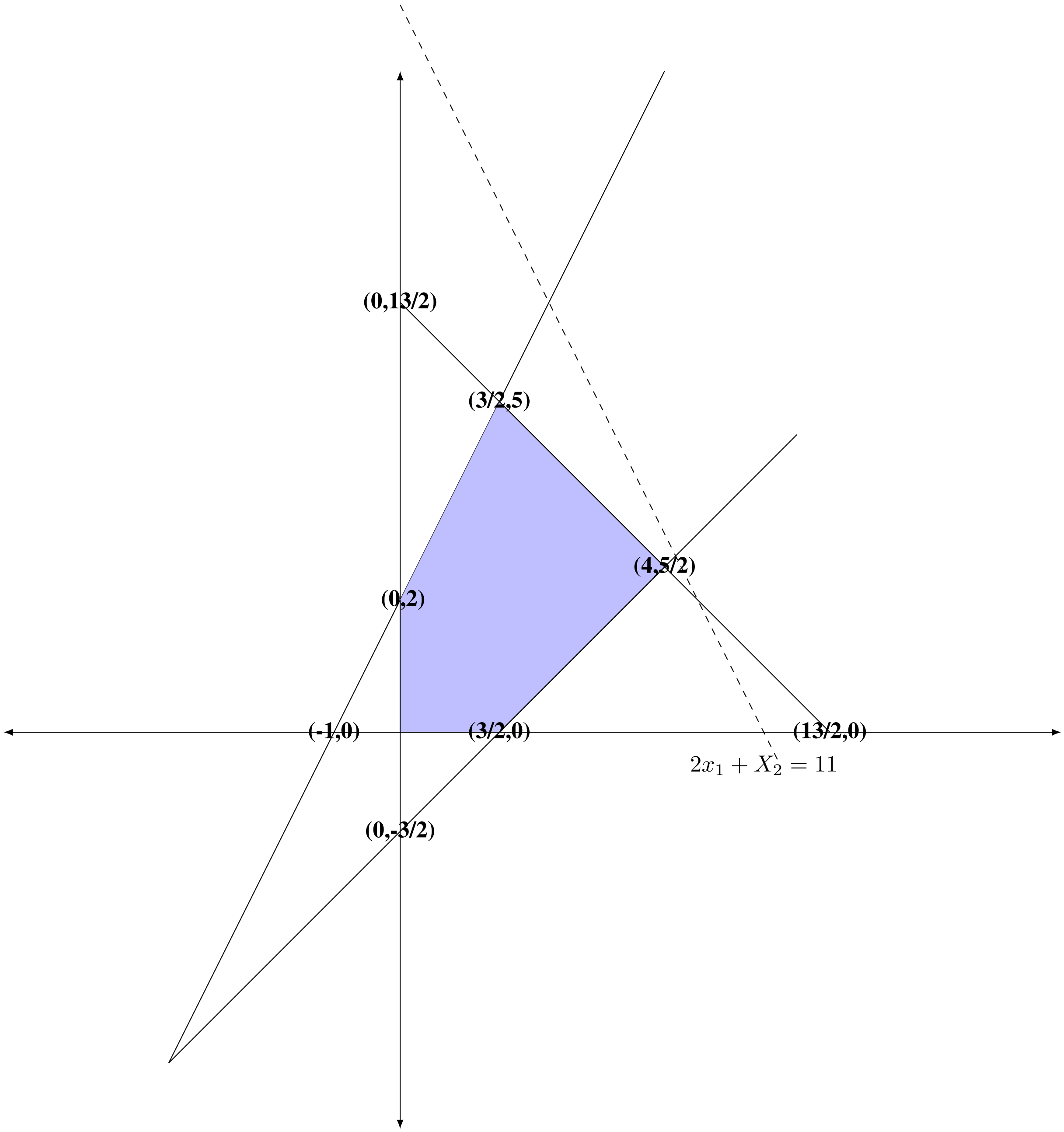
$$\begin{aligned} 2 \cdot x_1 - 2 \cdot x_2 &\leq 3 \\ -2 \cdot x_1 + x_2 &\leq 2 \\ 2 \cdot x_1 + 2 \cdot x_2 &\leq 13 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\text{ integer} \end{aligned}$$

(a) Solve the above problem graphically.

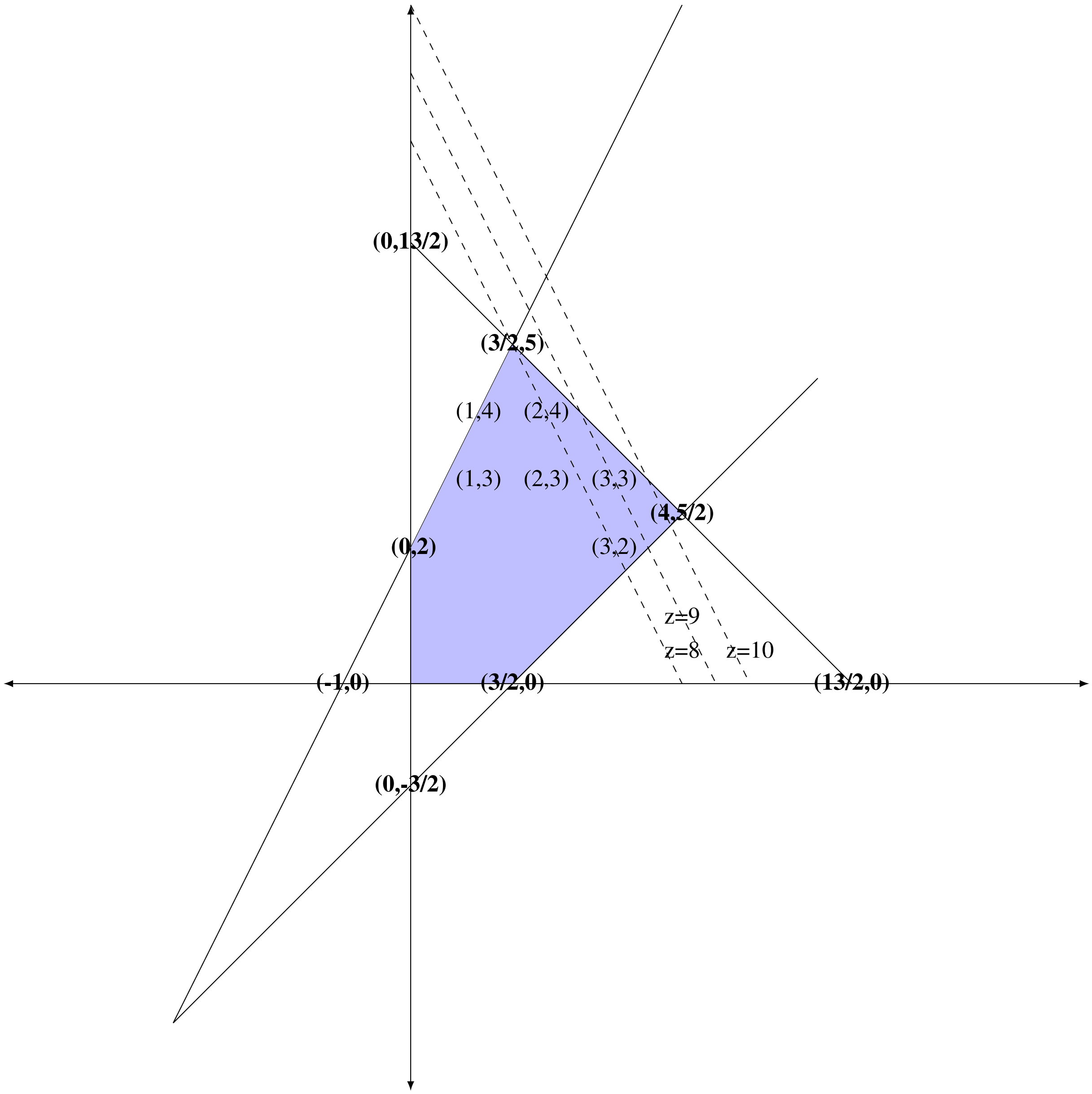
(b) Solve the above problem using the branch-and-bound technique discussed in class. All linear programming relaxations should be solved graphically.

Solution:

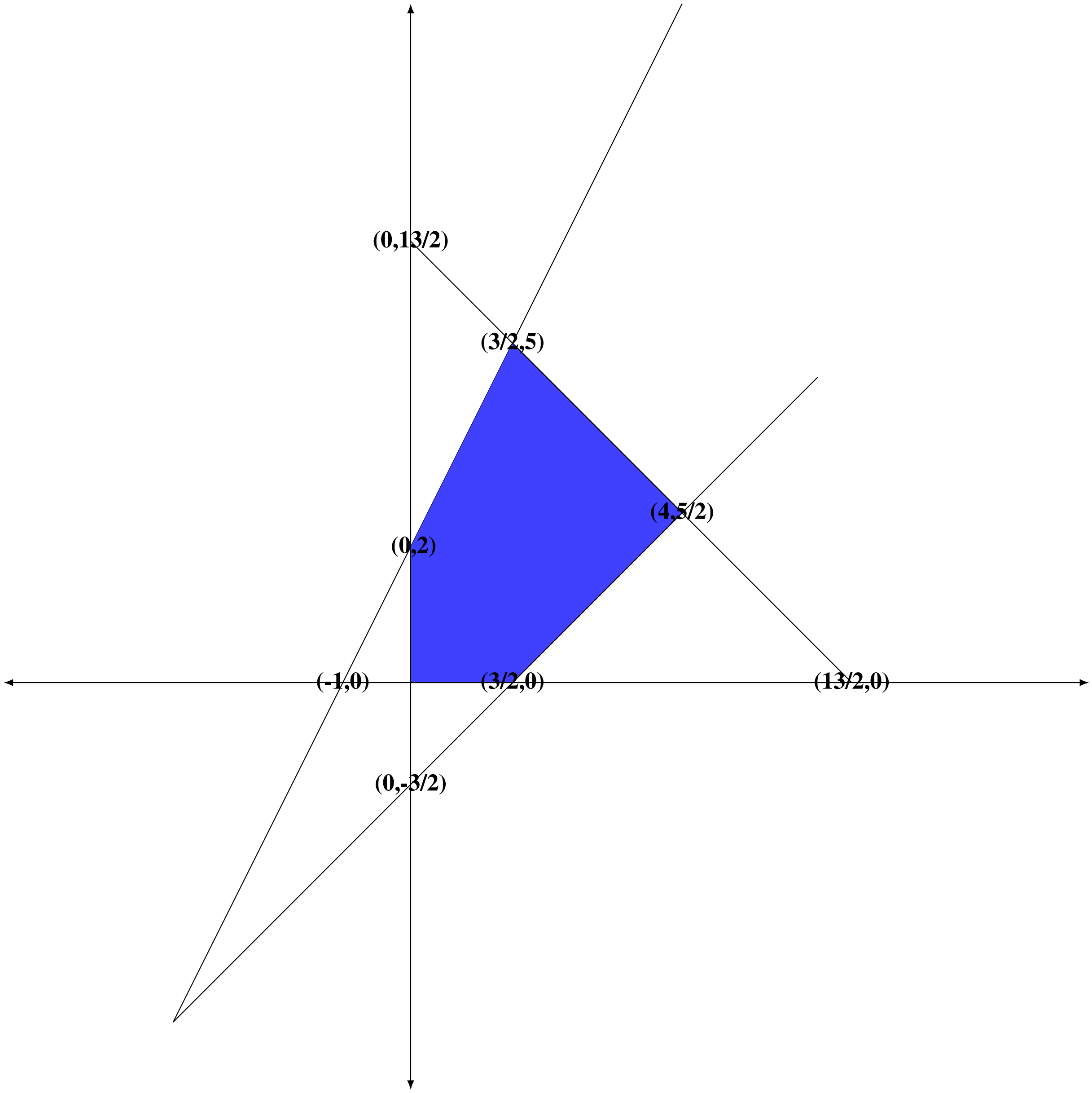
(a) If we were asked to solve the LP, then the optimal point would be a vertex. Since the objective function is $2 \cdot X_1 + X_2$, the result should be integer, too. That is why we can not use the point $(4, 5/2)$, $z = 10.5$ since it is not integer. Now we try to draw slip (skew) the objective function into the feasible solution to find the integral points (point), which lie on that line.



By testing the lines of $2X_1 + X_2 = \{10, 9, 8\}$, we find that the objective function $2 \cdot X_1 + X_2 = \{10\}$ does not contain any integer point in the feasible solution, so we try $2 \cdot X_1 + X_2 = \{9\}$, which contains the point $(3, 3)$ giving the optimal value $Z = 9$. These steps are shown in the following figure.

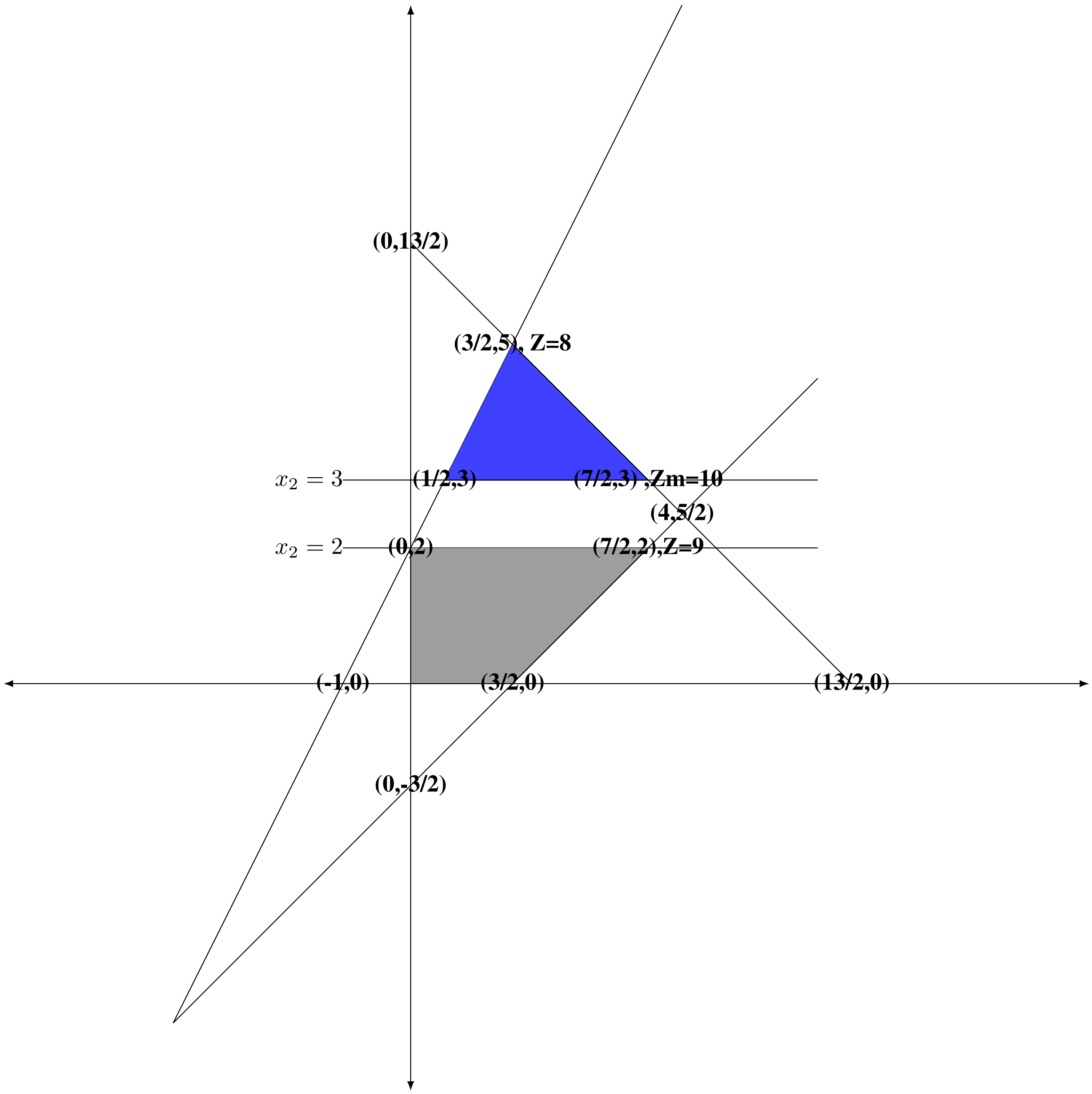

(0,13/2)
(3/2,5)
(1,4)
(2,4)
(1,3)
(2,3)
(3,3)
(4,5/2)
(0,2)
(3,2)
z=9
z=8
z=10
(-1,0)
(3/2,0)
(13/2,0)
(0,-3/2)

(b) In the following figure constraints and feasible solutions are shown.

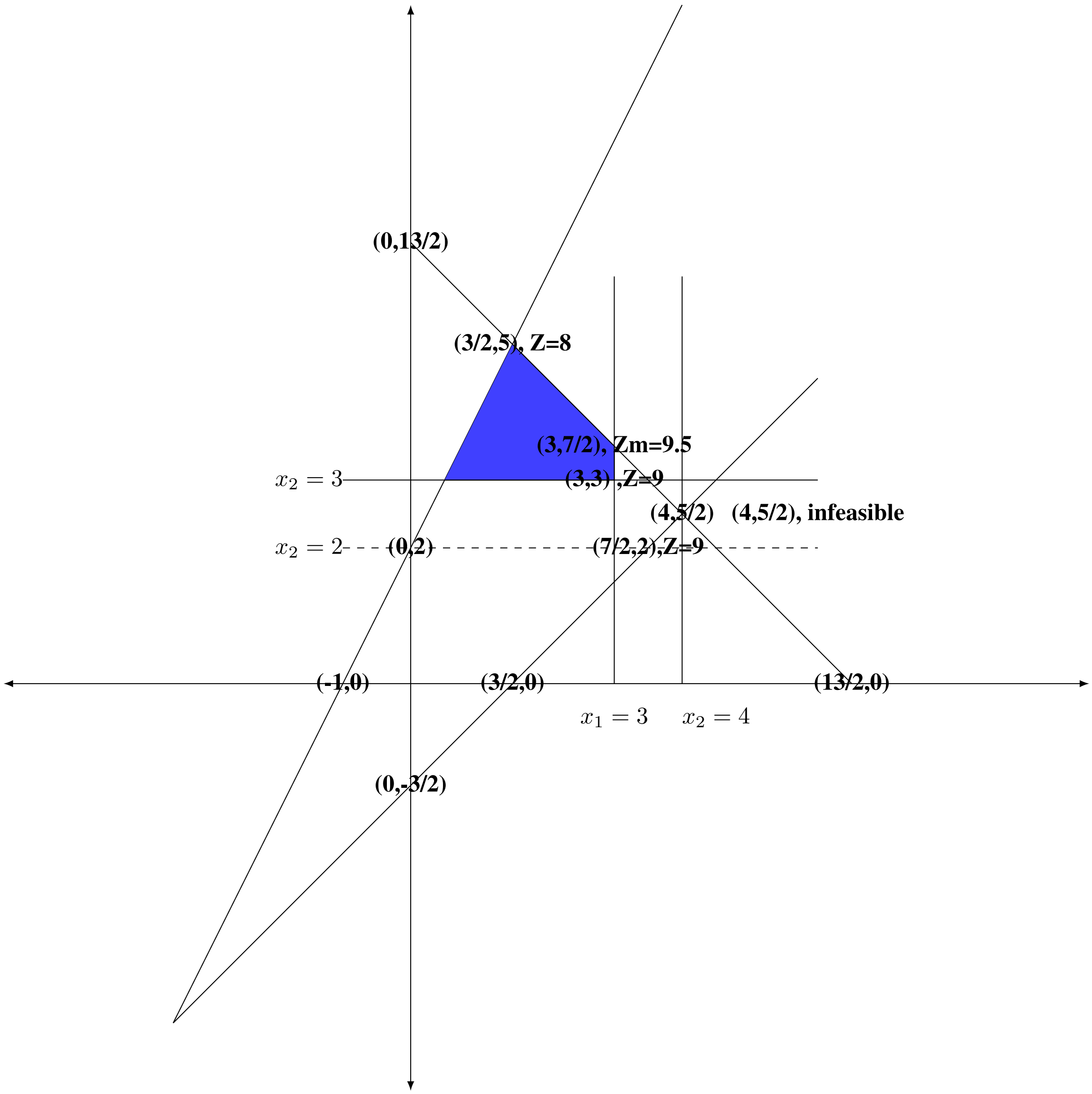


Since the coefficients in the objective function are integers and positive, one of the vertices $(4, 5/2)$ or $(3/2, 5)$ should be the answer for the linear programming. Since we have integer programming here we need to find out the integer numbers around it. for $(4, 5/2) \Rightarrow z = 10.5$, for $(3/2, 5) \Rightarrow z = 8$.

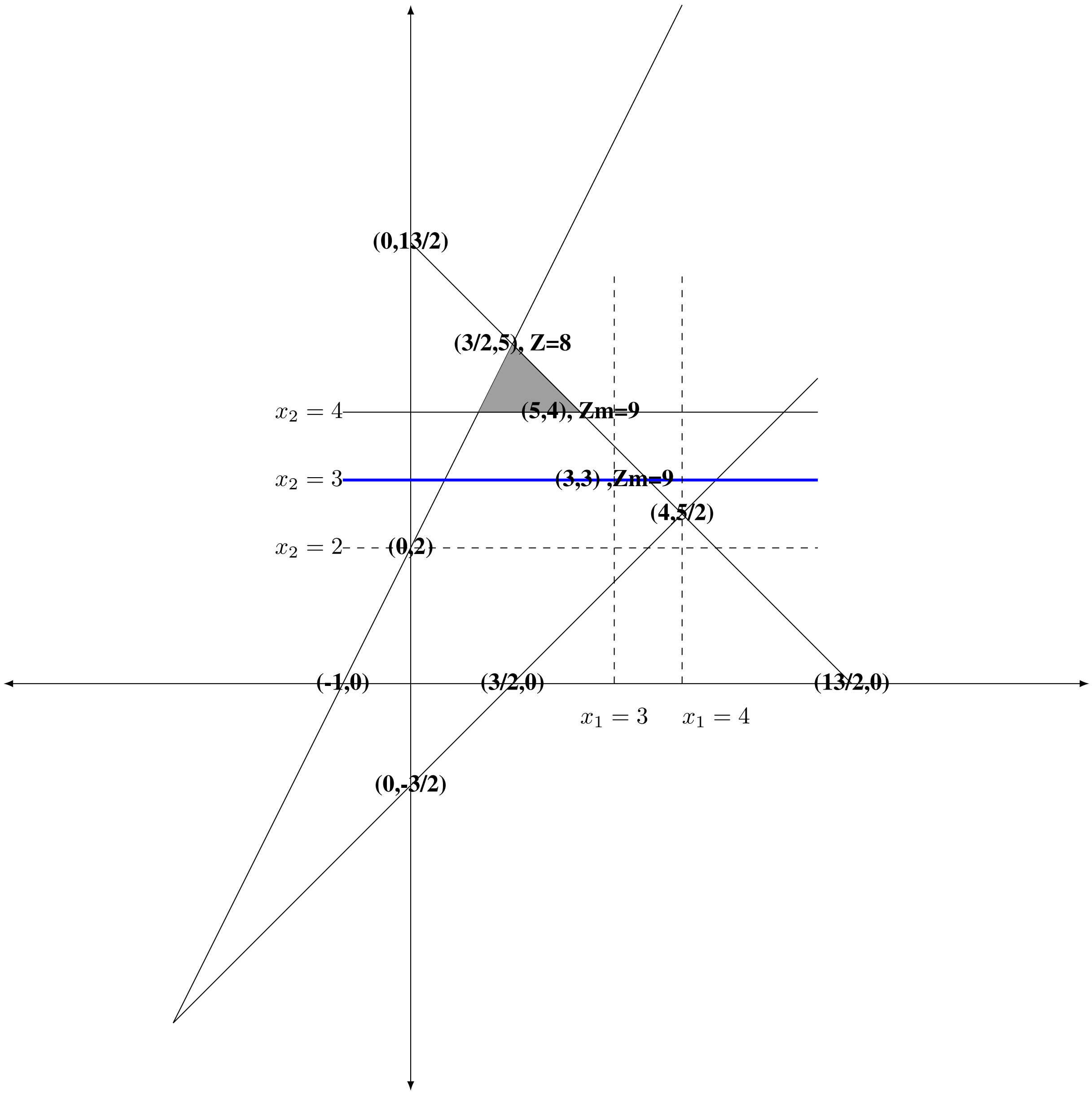
Because the first point has larger value, we try to use that by adding two constraints to it by $X_2 \leq 2$ $X_2 \geq 3$. Now we have two areas and again we should find the optimal points by checking the vertices.



So, now we find that $Z_{max} = Z_m = 10$ in the blue region, but still this is not an integer solution. So again we need to add two more constraints. Since the optimal vertex is $(7/2, 3)$, we add $X_1 \geq 4$ and $X_1 \leq 3$.



Again we see here that $Z_m = 9.5$ is not integer so we need to add two more constraints to the previous figure two: $X_2 \geq 4$ and $X_2 \leq 3$.



Here we see that $Z_m = 9$, and the corresponding vertex is $(3, 3)$.

Final solution : $(x_1 = 3, x_2 = 3) \Rightarrow (Z_{max} = 9)$.

□

4. The following problem is called the stagewise shortest path problem:
   You are given n cities, which are partitioned into $(N + 1)$ stages. City O is the only city in Stage 0 and city D is the only city in Stage N. Each city in stage k can advance to any city in stage $k + 1$, for $k = 0, 1, 2, \ldots (N - 1)$. The distance between city i and city j is denoted by d_{ij}.

   Formulate the dynamic programming recursion for the stagewise shortest path problem.

   Solution:

   Let $v_i, i = 1 \ldots m$ denote the i^{th} city with $v_1 = O$ and $v_m = D$. Also, let V_k denote the set of cities in stage k. Thus, $V_0 = \{v_1\}$ and $V_N = \{v_m\}$. For all cities $v_i \in V_k$ let $D(i, k)$ represent the length of the stagewise shortest path from v_1 to $v_i \in V_k$. Thus, we can define D as follows:

- $D(1,0) = 0$.
- For each $v_i \in V_k$, $D(i,k) = \min_{v_j \in V_{k-1}}(D(j,k-1) + d_{ij})$.

The goal is to find $D(m,N)$. □

5. Compute the value of an American put option on a stock with current price equal to \$100, strike price equal to \$98, and expiration date five weeks from today. The yearly volatility of the logarithm of the stock return is $\sigma = 0.30$. The risk-free interest rate is 4%. Use a binomial lattice with $N = 5$.

Solution: On page 246 of the textbook, this problem is solved for the case $N = 4$. Since the values of all parameters are the same, we are going to have the same recursions as in the textbook. They are the following:

$$v(N,j) = \max\{c - u^j \cdot d^{N-j} \cdot S_0, 0\},$$

when $j = 0, 1, ..., N$, and

$$v(k,j) = \max\{\frac{1}{R} \cdot [p_u \cdot v(k+1,j+1) + p_d \cdot v(k+1,j)], c - u^j \cdot d^{k-j} \cdot S_0\}.$$

$k = N, N-1, ..., 0$ and $j = 0, 1, ..., k$. By taking $N = 5$, and using the values of the constants from the textbook, one can get the following values for $v(k,j)$, and in particular, for $v(0,0)$. The calculations are done via a program written on $C++$.

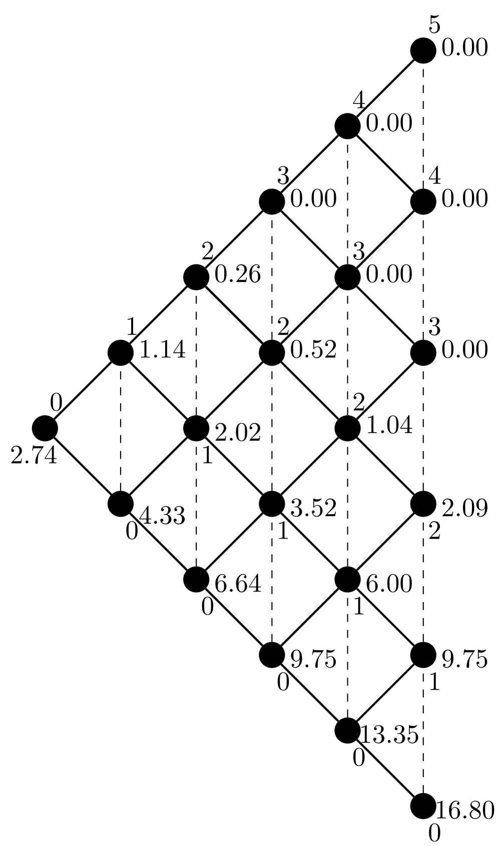


Figure 1: The values of $v(k,j)$

□