# CS 491 G Combinatorial Optimization

## Lecture Notes

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## 1. Linear Programming Problems

**A Product-Mix Problem:**

HiTech, inc., a small manufacturing firm produces two microwave switches, switch A and B. The return per unit of Switch A is \$5, whereas the return per unit of Switch B is \$8. Because of contractual commitments, HiTech must manufactures at least 20 units of Switch A per week, and based on the present demand for its products, it can sell all that it can manufacture. However, it wishes to maximize profit while determining the production sizes to satisfy various limits resulting from a small production crew. These includes

Assembly hours: 40 hours available per week
Testing hours: 30 hours available per week

Switch A requires 3 hours assembly and 2 hours of testing, Switch B requires 4 hours assembly and 1 hours of testing

**Determination of the decision variables**

The problem is obviously to determine the optimal number of each type of switch to manufacture based on the limited resources available. The variables directly under Hitech's control are

$x_1 =$ amount of Switch A manufactured per week
$x_2 =$ amount of Switch B manufactured per week

**Formulation of the objective**

The overall objective is to maximize weekly profit and because the unit returns for switches A and B are \$5 and \$8, respectively, the objective can be written as follows:

$$\text{maximize } z = 5x_1 + 8x_2 \quad \text{(profit per week)}$$

**Formulation of the constraints**

Based on the consumption rates of the two switches and the limited resources available, the production constraints can be formulated as follows:

$$3x_1 + 4x_2 \leq 40 \quad \text{(assembly hours per week)}$$
$$2x_1 + x_2 \leq 30 \quad \text{(testing hours per week)}$$

Also, the minimum requirement for Switch A is given simply as

$$x_1 \geq 20 \quad \text{(Switch A demand per week)}$$

The nonnegative restrictions on each variable are written as

$$x_1 \geq 0$$
$$x_2 \geq 0$$

As a result, the mathematical model for this problem may be summarized as follows:

$$\text{maximize } z = 5x_1 + 8x_2$$

subject to

$$3x_1 + 4x_2 \leq 40$$
$$2x_1 + x_2 \leq 30$$
$$x_1 \geq 20$$
$$x_2 \geq 0$$

**Max LP model canonical form (1)**

$$\max \ z = \vec{c}\vec{x}$$
$$\text{Constraints: } A\vec{x} \leq \vec{b}$$
$$\vec{x} \geq \vec{0}$$

For the above product-mix problem:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 5 & 8 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 \\ 3 & 4 \\ 2 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -20 \\ 40 \\ 30 \end{bmatrix}$$

**Max LP model standard form (2)**

$$\max\ z = \vec{c}\vec{x}$$
$$\text{Constraints: } A\vec{x} = \vec{b}$$
$$\vec{x} \geq \vec{0}$$

**Converting a linear program into standard form**

In general, it is far easier to deal with equations than with inequalities. In this section, we provide a simple technique that allows us to convert any inequality into an equation by means of introducing some additional variables into the formulation.

First, consider an inequality constraint set of the following form:

$$A\vec{x} \leq \vec{b}$$
$$\vec{x} \geq \vec{0}$$

We introduce a new variable $\vec{x}_s$, so that the above equation can be rewritten as

$$A[\vec{x}, \vec{x}_s] = \vec{b}$$
$$\vec{x} \geq \vec{0},\ \vec{x}_s \geq \vec{0}$$

Now it allows us to express an inequality in a more convenient equality format.

**Example 1.1:**

Given the constraints of the following linear programming model, covert it into standard form.

$$3x_1 + 4x_2 \leq 40$$

Introducing a new variable $x_3$, convert the inequality into standard form:

$$3x_1 + 4x_2 + x_3 = 40$$
$$x_3 \geq 0$$

**Definition: 1.1 Hyper-plane**

A linear equation $\vec{a}\vec{x} = c$ is called a hyper-plane in the n-dimensional Euclidean space $R^n$.

**Definition: 1.2 Half-space**

A linear relationship of the form $\vec{a}\vec{x} \le c$ is called a half-space in the n-dimensional Euclidean space $R^n$.

**Definition: 1.3 Convex set**

A set of $S$ is said to be convex if for any $\vec{x}_1, \vec{x}_2 \in S$
$\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \in S$, where $0 \le \lambda \le 1$

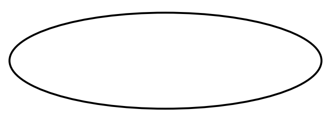

Figure 1. Illustration of a convex set

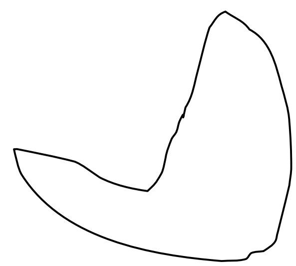

Figure 2. Illustration of a non-convex set

**Definition: 1.4 Polyhedral set**

A polyhedral set is the conjunction of a set of half spaces.

***Observation***: All Polyhedral sets are convex sets.

**Definition: 1.5 Linearly independent and dependent**

The vectors

$$\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n$$

in a vector space $V$ are said to be linearly independent if

$$k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n = 0$$

implies that all of the scalars $k_1 \quad k_2 \quad \ldots \quad k_n$ are equal to zero, otherwise the vectors are said to be linearly dependent.

**Example1.2:**

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$\vec{a}_1$ and $\vec{a}_2$ are linearly independent, $\vec{a}_1$ and $\vec{a}_3$ are not linearly independent, because

$$2\vec{a}_1 - \vec{a}_3 = \vec{0}$$

**Definition: 1.6 Basis**

Given an $m \times n$ matrix A ($m < n$), a collection of $m$ linearly independent columns is called a basis.

**Basic Solution**

Given

$$A\vec{x} = \vec{b}$$

We partition $A$ into

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n] = [B \quad N]$$

Likewise, we partition $\vec{x}$ into

$$\vec{x} = \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix}$$

$\vec{x}_B$ is the set of basic variables and $\vec{x}_N$ is the set of non-basic variables. Substituting the above partitioned expressions of $A$ and $\vec{x}$ into $A\vec{x} = \vec{b}$, we get

$$[B \quad N] \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = \vec{b}$$

Expanding the above equation, we have

$$B\vec{x}_B + N\vec{x}_N = \vec{b}$$

The inverse of matrix $B$ exists since $B$ is a collection of $m$ linearly independent vectors. Multiplying the above equation by $B^{-1}$, we get:

$$\vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N$$

Basic solution is obtained by setting the non-basic solution into zero: The solution

$$\vec{x} = \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = \begin{bmatrix} B^{-1}\vec{b} \\ \vec{0} \end{bmatrix}$$

is called a basic solution, with vector $\vec{x}_B$ called the vector of basic variables, and $\vec{x}_N$ is called the vector of non-basic variables. If, in addition, $\vec{x}_B = B^{-1}\vec{b} \geq 0$, then

$$\vec{x} = \begin{bmatrix} B^{-1}\vec{b} \\ \vec{0} \end{bmatrix}$$

is called a basic feasible solution (b f s) of the constraint system.

**Lemma 1.1:**

Max $\vec{c}\vec{x}$ is always attained at a basic feasible solution.
Proof is given in Ref. 2.

The total number of possible basic solution is $\begin{pmatrix} n \\ m \end{pmatrix}$.

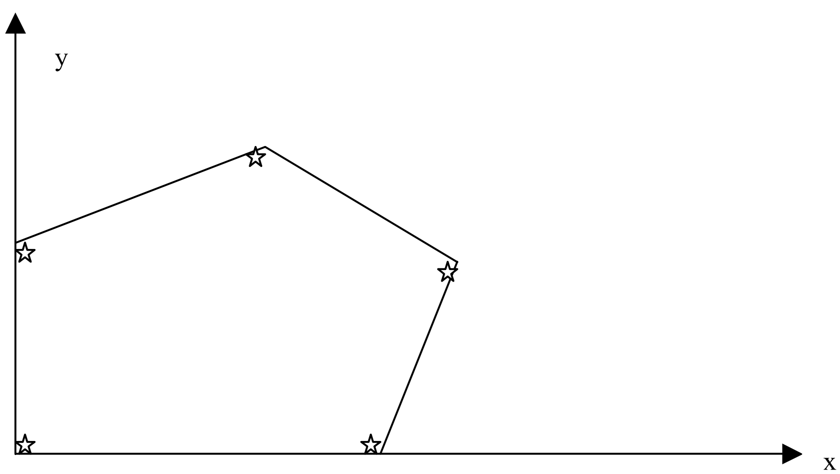


Figure 1. Illustration of basis points

**Algebra of the simplex method**

Now consider the objective function $z = \vec{c}\vec{x}$. Partition the cost vector $\vec{c}$ into basic and non-basic components

$$\vec{c} = (\vec{c}_B, \vec{c}_N)$$

Inserting the partitioned expression of $\vec{c}$ and $\vec{x}$ into $z = \vec{c}\vec{x}$, the objective function can be recast as

$$z = \vec{c}_B \vec{x}_B + \vec{c}_N \vec{x}_N$$

Since $\vec{x}_B = B^{-1}\vec{b} - B^{-1}N\vec{x}_N$, $z$ can be rewritten as

$$z = \vec{c}_B B^{-1}\vec{b} - (\vec{c}_B B^{-1} N - \vec{c}_N)\vec{x}_N$$

Let J = {Index set of non-basic variables}, then objective function can be rewritten as follows:

$$z = \vec{c}_B B^{-1}\vec{b} - \sum_{j=J} (\vec{c}_B B^{-1}\vec{a}_j - c_j) x_j$$

**Checking for optimality**

Observe that the coefficient $-(\vec{c}_B B^{-1}\vec{a}_j - c_j)$ of $x_j$ represents the rate of change of $z$ with respect to the non-basic variable $x_j$. That is,

$$\frac{\partial z}{\partial x_j} = -(\vec{c}_B B^{-1}\vec{a}_j - c_j)$$

Thus, if $\frac{\partial z}{\partial x_j} > 0$, then increasing the non-basic variable $x_j$ will increase $z$. The quantity $(\vec{c}_B B^{-1}\vec{a}_j - c_j)$ is sometimes referred as the reduced cost and for convenience is usually denoted by $z_j - c_j$. We can thus state the optimality conditions for a maximization linear programming problem.

**Optimality Conditions**

The basic feasible solution represented by $\vec{x} = \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = \begin{bmatrix} B^{-1}\vec{b} \\ \vec{0} \end{bmatrix} \geq 0$ will be optimal to (LP) if

$$\frac{\partial z}{\partial x_j} = -(z_j - c_j) = -(\vec{c}_B B^{-1}\vec{a}_j - c_j) \leq 0 \text{, for all } j \in J$$

or, equivalently, if

$$z_j - c_j = \vec{c}_B B^{-1}\vec{a}_j - c_j \geq 0 \text{, for all } j \in J$$

Note that because $z_j - c_j = 0$ for all basic variables, then the optimality conditions could also be stated simply as $z_j - c_j \geq 0$, for all $j = 1, \cdots, n$.

**Determining the entering variable**

Suppose there exists some non-basic variable $x_k$ with a reduced cost $z_k - c_k < 0$. Then $\frac{\partial z}{\partial x_k} > 0$ and the objective function can be improved by increasing $x_k$ from its current value of zero. Typically, we choose to increase that non-basic variable that forces the greatest rate of change of the objective, that is, the non-basic variable with the most negative $z_j - c_j$. The selected variable $x_k$ is called the entering variable. That is, $x_k$ is going to enter the current basis.

**Determining the leaving variable**

Let's now investigate the consequence of the preceding results. The question is: we want to bring in $\vec{a}_k$ into the basis, which variable (column) should be kicked out? Consider the vector of coefficient of the non-basic variable $x_k$ and let

$$\vec{\alpha}_k = B^{-1}\vec{a}_k$$

Note that the rate of change of the basic variables with respect to the non-basic variable $x_k$ is given by

$$\frac{\partial x_B}{\partial x_k} = -B^{-1}\vec{a}_k = -\alpha_k$$

That is, if the non-basic variable $x_k$ is increased from its current value of zero while holding all other non-basic variable at zero, the basic variables will change according to the relationship

$$x_B = B^{-1}b + x_k(-B^{-1}\vec{a}_k) = B^{-1}b - x_k\alpha_k$$

Because all variables must remain nonnegative, it follows that

$$x_B = B^{-1}b - x_k\alpha_k \geq 0$$

Now let

$$\vec{\beta} = B^{-1}\vec{b}$$

Since $\vec{x}_\beta = \vec{\beta} - x_k\vec{\alpha}_k \ge 0$, an upper bound on $x_k$ can be easily found quite easily as

$$\vec{x}_k \le \min\left\{\frac{\vec{\beta}_i}{\vec{\alpha}_{i,k}} \quad \vec{\alpha}_{i,k} > 0\right\}$$

This process provides a very simple method for determining the maximum value of the entering variable.

**Algorithm 1.1**

1. Represent matrix *A* in the following way $A = (B \quad N)$
2. Assume a feasible basis *B*.
3. If *B* is optimal, then stop.
4. Bring in the variable with the maximum $\frac{\partial z}{\partial x_j}$.
5. Kick out the variable with respect to column that drops to zero.
6. Go to step 3 and check new basis optimal or not.

**Example 1.3:**

$$\max \ z = 2x_1 + 3x_2$$

Constraints:

$$x_1 - 2x_2 \le 4$$
$$2x_1 + x_2 \le 18$$
$$x_2 \le 10$$
$$x_1 \ge 0 \quad x_2 \ge 0$$

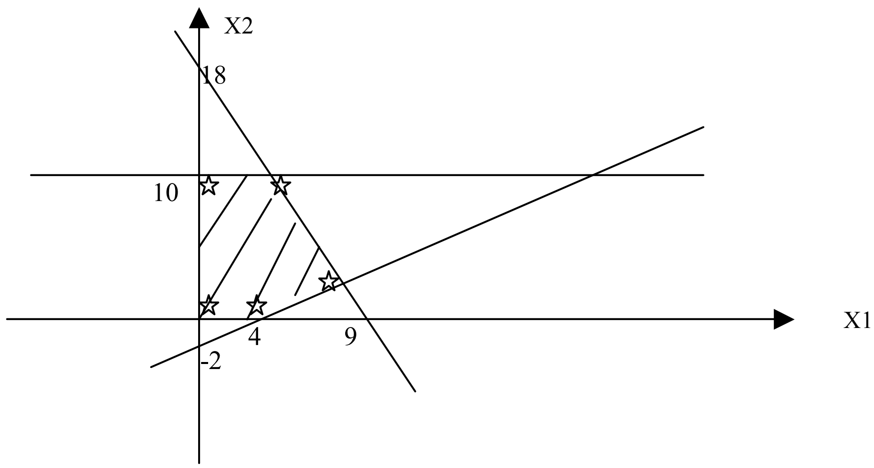


Figure 2. Solution by a graph for this example

From the above graph, we can get every corner point coordinates, then substitute into the maximum equation, and we get the solution of $z = 38$.

Using our algorithm

$$
\begin{gathered}
x_1 - 2x_2 + x_3 = 4 \\
2x_1 + x_2 + x_4 = 18 \\
x_2 + x_5 = 10
\end{gathered}
$$

$$x_1 \geq 0 \qquad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0 \quad x_5 \geq 0$$

The data for this problem can be summarized as follows:

$$\vec{c} = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix} \; A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 18 \\ 10 \end{bmatrix}$$

We begin the solution process by choosing a convenient starting basis matrix *B*. Because the solution will be determined by $B^{-1}$, we will choose the starting basis matrix *B* = *I*.

$$\vec{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{\partial z}{\partial x_1} = 2 \qquad\qquad \frac{\partial z}{\partial x_2} = 3$$

Because $\frac{\partial z}{\partial x_2} > \frac{\partial z}{\partial x_1}$, therefore choose $x_2$ as the entering variable.

$$\begin{aligned} x_3 &= 4 - x_1 + 2x_2 \\ x_4 &= 18 - 2x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

By the minimum ratio test, the maximum value of $x_2$ is equal to minimum {18, 10) = 10. Therefore, $x_5$ is the departing variable. Then we have:

$$\begin{aligned} x_2 &= 10 - x_5 \\ x_3 &= 24 - x_1 + 2x_5 \\ x_4 &= 8 - 2x_1 + x_5 \end{aligned}$$

$$z = 2x_1 + 30 - 3x_5$$

This solution is not yet optimal because $\frac{\partial z}{\partial x_1} = 2 > 0$, thus $x_1$ is chosen as the entering variable. As before the minimum ratio test yields minimum {24, 4) = 4, and $x_4$ is the departing variable. Then we have:

$$\begin{aligned} x_1 &= 4 - \frac{1}{2}x_4 + \frac{1}{2}x_5 \\ x_2 &= 10 - x_5 \\ x_3 &= 20 + \frac{1}{2}x_4 - \frac{3}{2}x_5 \end{aligned}$$

$$z = 38 - x_4 - 2x_5$$

Therefore, maximum value is 38, at point of (4,10).

**Duality**

Given primal (P)

$$\begin{gathered} \text{max } z = \vec{c}\vec{x} \\ \text{Constraints: } A\vec{x} \le \vec{b} \end{gathered}$$

$$\vec{x} \geq \vec{0}$$

we define the dual to be

$$\begin{gathered} \min\ w = \vec{b}\vec{y} \\ \text{Constraints: } \vec{y}A \geq \vec{c} \\ \vec{y} \geq \vec{0} \end{gathered}$$

**Example 1.4:**

Write the dual of Example 1.3.

$$\begin{gathered} \min\ 4y_1 + 18y_2 + 10y_3 \\ y_1 + 2y_2 \geq 2 \\ -2y_1 + y_2 + y_3 \geq 3 \\ y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0 \end{gathered}$$

So

$$B = \begin{bmatrix} 4 \\ 18 \\ 10 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 2 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

**Lemma 1.2**

Dual of dual is primal. The proof is given in Ref. 2.

**FORMULATION OF THE LINEAR PROGRAMMING DUAL**

**The canonical form of the dual**

The basic characteristics of the canonical primal and dual are summarized in table 1.1. As we shall see when discussing general duality, these relationships can be extended to any linear programming problem.

TABLE 1.1 CANONICAL PRIMAL-DUAL RELATIONSHIPS

| Maximization problem | Minimization problem |
|---|---|
| m constraints | m variables |
| $\leq$ | $\geq 0$ |
| n variables | n constraints |
| $\geq 0$ | $\geq$ |

**General duality**

The objective may be either of a maximizing or minimizing form, variables may be restricted or unrestricted, or the constraints may be of any form and any mixture of the forms. These motivate the rules for finding the dual from the general form. By utilizing the relationship in Table 1.2, it is possible to write the dual problem for a given linear programming without going through the intermediate step of transforming the problem to canonical form.

TABLE 1.2 PRIMAL-DUAL RELATIONSHIPS

| Maximization problem | Minimization problem |
|---|---|
| Constraints | Variables |
| $\le$ | $\ge 0$ |
| $\ge$ | $\le 0$ |
| $=$ | unrestricted |
| Variables | Constraints |
| $\ge 0$ | $\ge$ |
| $\le 0$ | $\le$ |
| unrestricted | $=$ |

**Example 1.5:**

$$\max\ z = \vec{c}\vec{x}$$
$$\text{Constraints: } A\vec{x} = \vec{b}$$
$$\vec{x} \ge \vec{0}$$

$$\min\ z = \vec{b}\vec{y}$$
$$\text{Constraints: } \vec{y}A \ge \vec{c}$$
$$\vec{y}\ \text{unrestricted}$$

**Weak Duality:**

Given (P) & (D)

$$\vec{c}\vec{x} \le \vec{b}\vec{y}$$

***Proof***: $\vec{c}\vec{x} \le \vec{y}(A\vec{x}) \le \vec{y}\vec{b} \le \vec{b}\vec{y}$

**Corollary**

If (P) is unbounded, (D) is infeasible.
If (D) is unbounded, (P) is infeasible.
If (P) is infeasible, (D) is unbounded or infeasible.

**Strong Duality:**

Given (P) & (D) and $\vec{x}^*$ optimal for (P) and $\vec{y}^*$ optimal for (D), then

$$\vec{c}\vec{x}^* = \vec{b}\vec{y}^*$$

***Proof:*** Given primal (P)

$$\text{Max } z = \vec{c}\vec{x}$$
$$\text{Constraints: } A\vec{x} \le \vec{b}$$
$$\vec{x} \ge \vec{0}$$

we can rewrite (P) as :

$$\text{Max } z = \vec{c}\vec{x} + \vec{0}\vec{x}_S$$
$$\text{Constraints: } [A, I]\begin{bmatrix} \vec{x} \\ \vec{x}_S \end{bmatrix} = \vec{b}$$
$$\vec{x} \ge \vec{0},\ \vec{x}_S \ge \vec{0}$$

Let $B$ be the optimal basis. Then,

$$z = \vec{c}_B B^{-1}\vec{b},\ \vec{x}^* = B^{-1}\vec{b}$$

The Dual (D) is:

$$\text{Min } \vec{y}\vec{b}$$
$$\text{Constraints: } \vec{y}A \ge \vec{c}$$
$$\vec{y} \ge \vec{0}$$

Choose $\vec{y}^* = \vec{c}_B B^{-1}$ then we have $\vec{c}\vec{x}^* = \vec{y}^*\vec{b}$. So we have a vector $\vec{y}^*$ in the dual space, where the dual objective function has the same value as the primal objective function. All we need to show is that $\vec{y}^*$ is feasible.
Note that the primal basis must satisfy $z_j - c_j \ge 0$ for all $j$. Thus, we have

$$\vec{c}_B B^{-1}A - \vec{c} \ge 0$$

i.e. $\vec{y}^* A \ge \vec{c}$. Since, we also have $\vec{c}_B B^{-1}I - \vec{0} \ge \vec{0}$, i.e. $\vec{c}_B B^{-1} \ge \vec{0}$, we have:

$$\vec{y}^* \ge \vec{0}$$

Thus, we have a dual feasible solution, such that the primal and dual objective functions have the same value. This proves the Strong Duality Theorem.

**Farkas Lemma**

Either $\exists \vec{x} \quad A\vec{x} \le \vec{b}$, $\vec{x} \ge \vec{0}$ or $\exists \vec{y} \ge \vec{0} \quad \vec{y}A \ge \vec{0}$, $\vec{y}\,\vec{b} < 0$

***Proof***: Consider the linear program (P):

$$\text{Max } z = \vec{0}\vec{x}$$
$$A\vec{x} \le \vec{b}$$
$$\vec{x} \ge \vec{0}$$

Its dual is (D):

$$\text{Min } \vec{b}\vec{y}$$
$$\vec{y}A \ge \vec{0}$$
$$\vec{y} \ge \vec{0}$$

Case 1: (P) and (D) have finite optimal (D) must have optimum zero, so we have

$$\vec{b}\vec{y} = 0$$

Case 2: (P) is unbounded, (D) is infeasible. This is not possible, since (D) is always feasible ( $\vec{y} = \vec{0}$ !)
Case 3: (D) is unbounded, (P) is infeasible. Since (D) is unbounded, there is a vector $\vec{y}$, such that $\vec{b}\vec{y}$ tends to $-\infty$. From the continuity of the solution space, we know that there must exist a vector $\vec{y}$, such that $\vec{b}\vec{y} < 0$.
Case 4: (P) and (D) are both infeasible. This is not possible because $\vec{y} = \vec{0}$ satisfies (D).

**Complementary Slackness**

Given a primal and dual optimal pair $(\vec{x}^*, \vec{y}^*)$, let $\vec{s}^* = \vec{b} - A\vec{x}^*$ and $\vec{t}^* = \vec{y}^* A - \vec{c}$, then

$$\vec{s}^*\vec{y}^* = 0$$
$$\vec{t}^*\vec{x}^* = 0.$$

***Proof:***

$$\vec{c}\vec{x}^* = (\vec{y}^* A - \vec{t}^*)\vec{x}^*$$
$$= \vec{y}^* A\vec{x}^* - \vec{t}^*\vec{x}^*$$
$$= \vec{y}^*(\vec{b} - \vec{s}^*) - \vec{t}^*\vec{x}^*$$
$$= \vec{y}^*\vec{b} - \vec{y}^*\vec{s}^* - \vec{t}^*\vec{x}^*$$

So we have $\vec{y}^*\vec{s}^* + \vec{t}^*\vec{x}^* = 0$. But all the vectors, $\vec{s}^*, \vec{y}^*$, $\vec{t},^* \vec{x} \geq \vec{0}$. The theorem follows.

## References


1. William Cook, William H. Cunningham, William R. Pulleyblank, and Alexander Schrijver. *Combinatorial Optimization*. John Wiley & Sons, Inc., 1998.
2. James P. Ignizio and Tom M. cavalier. *Linear Programming*. Prentice Hall, Inc., Upper Saddle River, New Jersey 07458, 1994.