Final Solutions

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1. Using the Simplex procedure, it is easy to see that the optimal value of the primal is 12 and it is obtained at $(x_1, x_2) = (0, 4)$. The dual of the input system is:

$$z = \min \ 8y_1 + 6y_2 + 4y_3$$
$$s.t.$$
$$y_1 + 2y_3 \geq 2 \quad (1)$$
$$y_1 + y_2 + y_3 \geq 3 \quad (2)$$
$$\vec{y} \geq 0$$
$$(3)$$

Let $\vec{y^*}$ denote the optimal dual vector. Observe that in the primal system, the first two relationships are not met with equality; this forces y_1^* and y_2^* to be 0, using the Complementary Slackness Theorem. Further, at optimality, $8y_1 + 6y_2 + 4y_3 = 12$, which implies that $y_3^* = 3$. Thus, the dual values are $y_1^* = 0, y_2^* = 0, y_3^* = 3$.

2. Let us concentrate on tree flows, since we know that if the min-cost flow problem has a feasible tree solution, then it has a feasible tree solution. First note that if a feasible solution exists, then the Network Simplex (NS) procedure cannot terminate with an artificial arc carrying positive flow, since we can always replace this arc by a path of length at most $n-1$ proper arcs (i.e. arcs of the input network) and having cost at most $(n-1)(\max |c_e| : e \in E)$. In other words, the solution using the artificial arc is not optimal and hence NS would not terminate with such a solution. It follows that if the Network Simplex Method terminates with an artificial arc having positive flow, then the original problem does not have a feasible solution.

3. Given: A bipartite graph G with bipartition (P, Q). In order to make it a flow network, add a source vertex s and a sink vertex t. Insert directed arcs from s to all $p \in P$ and from all $q \in Q$ to t. Direct all arcs in G from P to Q. Finally, each edge in the new network is given capacity 1. We argued in class, that the maximum flow algorithm applied to this network solves the matching problem. Let the flow at some stage of the algorithm be $\vec{x}$. Without loss of generality, we assume that $\vec{x}$ is integral.

   Claim: 0.1 *There is a one-to-one correspondence between matchings and (integral) feasible flows.*

   <u>Proof</u>: *Any integral feasible flow will have some edges with flow 1 and the rest with flow 0, since the capacity constraints need to be obeyed. Ignoring the edges $s \rightsquigarrow p \in P$ and $q \in Q \rightsquigarrow t$, the edges with flow 1 represent a matching in G. Let us call this set $M(\vec{x})$ i.e. $M(\vec{x})$ represents the set of edges in G, which have flow 1, under the current flow $\vec{x}$.*

   Observe that no two edges in $M(\vec{x})$ can be incident to the same vertex $q' \in Q$, owing to the capacity constraint of the edge $q't$; likewise no two edges can originate from the same vertex $p' \in P$, without violating either the feasibility of sp' or the integrality of $\vec{x}$. □

   Consider the structure of an $M(\vec{x})$ augmenting path $\mathcal{P}$, which starts at vertex p and ends at vertex q. Clearly p and q are $M(\vec{x})-$exposed, as per the definition of an $M(\vec{x})-$augmenting path. It follows that all edges incident on p and q carry 0 flow. It also follows that if $p \in P$, then $q \in Q$ and *vice versa* (Check this out!). Without loss of generality, assume $p \in P$ and $q \in Q$. Clearly, one unit of flow can can be pushed from s to p, all through $\mathcal{P}$ and from q to t, without violating capacity constraints, i.e. we have an $\vec{x}-$augmenting path. The edges of the matching $M(\vec{x})$ will be reverse arcs in this augmenting path.

If there is an $\vec{x}-$augmenting path, in the network, it means that at least one unit of flow can be pushed from s to t without violating capacity constraints on the edges. Consider the structure of this augmenting path $\mathcal{P}$. The edge from s to $p \in P$ must be a forward arc and the edge from $q \in Q$ to t must be forward arcs; it follows that they carried zero flow initially. Thus, initially no arc incident on p carried positive flow; likewise for q, in other words p and q are $M(\vec{x})$ exposed under the matching corresponding to flow $\vec{x}$. Further every alternate edge in the path $\mathcal{P}$ (excluding sp and qt) must be carrying zero flow, starting from p and ending at q. Thus, we have an $M(\vec{x})$-augmenting path, as desired.

You can complete the rest yourself!