Termination of the Network Simplex Method

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In class, we discussed the Simplex Method for solving the Minimum-Cost Flow problem. The key idea was the recognition of trees of the input network as the *basic solutions* of the system. Theorem (4.8) is a restatement of the theorem we derived for general linear programs, i.e. that if a linear program is feasible, then it has a basic feasible solution and if it has an optimal solution, then it has a basic feasible optimal solution.

The problem with the Simplex Method is that degenerate solutions could lead to cycling, i.e. starting at a basis, we go through a sequnce of bases and then return to the basis at which we started. In terms of networks, we start from a tree solution T, go through trees T', T''... and then return to T. Cycling is an observed phenomenon and not merely a theoretical issue; hence it needs to be addressed. (See Exercise 4.24 in [CCPS98]).

Definition: 0.1 *A solution T is said to be degenerate if at least one of the arcs in T carries 0 flow, as determined by the flow $\vec{\mathbf{x}}$.*

Definition: 0.2 *An iteration of the Simplex method that changes the tree but not the flow is said to be a degnerate iteration.*

Note that degenerate iterations can occur onli if the initial tree is degenerate.

Let us focus on a particular tree solution T, which has been rooted at vertex r.

Definition: 0.3 *An arc $h = pq$ is said to be away from r in T, if $p \in R(T, h)$; otherwise it said to be toward r in T.*

Recall that $R(T, h)$ is the set of vertices of the network that can be reached from r in T, without using the arc h.

Definition: 0.4 *A tree T is said to be strongly feasible if it determines a feasible flow, in which every arc h of T having $x_h = 0$ is away from r in T.*

Observe that a non-degenerate tree is trivially strongly feasible. Likewise, the tree with which we initialize the Network Simplex procedure is also strongly feasible.

In the Simplex Method, at each iteration, we choose an arc to be removed from the basis and another arc to be brought in. The arch that leaves is chosen according to the following rule:

Leaving Arc Rule: Choose h to be the first reverse arc of $C(T, e)$ with $x_h = 0$.

Once again recall that $C(T, e)$ is the cycle formed in T, when edge e is to be brought in. Further according to the rules discussed in selecting the cycle $C(T, e)$, there is a unique starting point s; consequently "first" arc is well defined. (See Figure 4.9 on Pg. 105. The second arc of $C(T, e)$ is the first reverse arc.)

Proposition: 0.1 *If T is strongly feasible and $\hat{T}$ is obtained from T, using the leaving arc rule, then $\hat{T}$ is strongly feasible.*

<u>Proof</u>: *Let e be the entering arc, h be the leaving arc, $\vec{\mathbf{x}}$, the old flow and $\hat{\vec{\mathbf{x}}}$ the new flow. Observe that edge changes and flow changes take place only in $C(T, e)$. The flows and orientations of all edges not in $C(T, e)$ are the same in $\hat{T}$ as in T. Hence we focus on the edges of $C(T, e)$ only.*

Consider the case in which the minimum flow on all reverse arcs, i.e. θ *is greater than* 0*. Then the arcs* g *of* $C(T,e)$ *having* $x_g = 0$*, are precisely those reverse arcs of* $C(T,e)$*, for which* $x_g = \theta$*. Since the first of these arcs is chosen to leave, the others will be away from* r *in* $\hat{T}$*. Why? Focus on Figure* 4.9*. If a reverse arc* $h = pq$ *is before* e *in* $C(T,e)$*, then* p *can be reached from* s *through* e*, without touching* q *i.e. it is in* $R(\hat{T},h)$*. If a reverse arc* $h = pq$ *is after* e *in* $C(T,e)$*, then its head must be away from* s *and hence* p *can once again be reached from* r *without using* q *and hence it is in* $R(\hat{T},h)$*.*

Now consider the case in which $\theta = 0$*. Let* $e = vw$ *and* s *be the first node of* $C(T,e)$ *(as in Figure* 4.9*). The arcs* g *of* $C(T,e)$ *for which* $\hat{x}_g = 0$*, are precisely those for which* $x_g = 0$ *(including* e*). But* T *is strongly feasible; hence arcs* g *other than* e *for which* $x_g = 0$ *must all be forward arcs of the paths in* T *from* s *to* v *and* s *to* w*. This means that none of the arcs in the path from* s *to* v *can be chosen; consequently the first reverse arc on the path from* w *to* s*, which is the same as the last forward arc on the path from* s *to* w *is chosen. Thus all the arcs* g *for which* $\hat{x}_g = 0$ *are indeed away from* r *in* $\hat{T}$ *and* $\hat{T}$ *is strongly feasible. (Note that* e *is a forward arc of* $C(T,e)$ *and is trivially away from* r *in* $\hat{T}$*.* □

Theorem: 0.1 *The Network Simplex Method started with a strongly feasible tree and using the leaving arc rule terminates finitely.*

Proof: *It suffices to show that cycling cannot occur. Suppose that a degnerate iteration takes* T *to* $\hat{T}$*, with associated path cost vectors* $\vec{\mathbf{y}}$ *and* $\hat{\vec{\mathbf{y}}}$*, i.e.* $\theta = 0$*. From the previous Proposition, we know that when* $\theta = 0$*, the leaving arc* h *is a forward arc of the path in* T*, from the first node of* $C(T,e)$ *to the head of the entering arc* e*. It follows that the head* w *of the entering arc* e *is not in* $R(T,h)$*, since we have to use the tail of* e *on the unique path from* r *to* w*. We can now use proposition* 4.10 *in [CCPS98] to conclude that* $\hat{y}_a \leq y_a$ *for all* $a \in V$*, while* $\hat{y}_w < y_w$*. (Although we are using the second condition of Proposition* 4.10*,* $\bar{c_e}$ *is negative; otherwise simplex will terminate!) It follows that* $\sum_{a \in V} \hat{y}_a \leq \sum_{a \in V} y_a$*. Hence any sequence of degenerate iterations, results in a strict decrease in the sum of the path costs; consequently no tree can repeat and the procedure terminates finitely.* □

References

[CCPS98] William Cook, William H. Cunningham, William Pulleyblank, and Alexander Schrijver. *Combinatorial Optimization.* John Wiley & Sons, 1998.