A Direct Approach to Spatial Averaging: Revisiting Baccelli's Result

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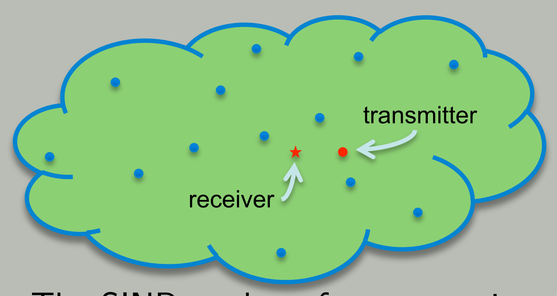
Problem: Computing the Spatially Averaged Outage Probability



M interferers are randomly distributed within an *arbitrary* region $\mathcal{A}$., where M can be *fixed* (e.g., BPP) or *random* (e.g., PPP).

- The SINR at the reference receiver is:

$$\gamma = \frac{g_0}{\mathsf{SNR}^{-1} + \sum_{i=1}^{M} I_i g_i r_i^{-\alpha}}$$

where:

$\{r_i\}$ are the distances to the M interferers.
$\{g_i\}$ are the fading gains (e.g., i.i.d. exponential).
$\{I_i\}$ are Bernoulli variables indicating interference ($\mathbb{P}[I_i = 1] = p$).
$\alpha \geq 2$ is the path-loss exponent.

- The goal is to find the spatially averaged outage probability:

$$\epsilon = \mathbb{E}_{\mathbf{g},\mathbf{r}} \left(\mathbb{P} \left[\gamma \leq \beta \right] \right)$$

where β is the SINR threshold.

- We would like to do this directly; i.e., without resorting to transforms such as a Laplace transform.

The Direct Approach to Spatial Averaging [1]

$$\epsilon(\mathbf{r}) = \mathbb{P}\left[\gamma \leq \beta \middle| \boldsymbol{r}\right]$$

Fix the r and find conditional outage probability by averaging over just the fading

$$\begin{aligned} \epsilon[M] &= \mathbb{E}_{\mathbf{r}} \left[\epsilon(\mathbf{r}) | M \right] \\ &= \int \epsilon(\mathbf{r}) f_{\mathbf{r}} \left(\mathbf{r} | M \right) d\mathbf{r} \end{aligned}$$

Fix M and find spatial average by taking expectation with respect to r.

If BPP, can stop, since M is fixed.

$$\begin{aligned} \epsilon &= \mathbb{E}_M \left[\epsilon[M] \right] \\ &= \sum_{m=0}^{\infty} p_M[m] \epsilon[m] \end{aligned}$$

If M is random, then take expectation with respect to the M.

As a final step, can take the limit as the area extends to the infinite plane.

Specific Cases

- In Rayleigh fading [2, 3],

$$\epsilon(\mathbf{r}) = 1 - e^{-\beta/\mathsf{SNR}} \prod_{i=1}^{M} \left(1 - p + \frac{p r_i^{\alpha}}{\beta + r_i^{\alpha}} \right).$$

- When the M interferers are i.u.d. over $\mathcal{A}$ (i.e., a binomial point process):

$$\epsilon[M] = 1 - e^{-\beta/\mathsf{SNR}} \left(1 - p + p \int f_r(r) \frac{r^{\alpha}}{\beta + r^{\alpha}} dr \right)^{M}$$

where the integral may be computed for an arbitrarily distributed r as

$$\int f_r(r) \frac{r^{\alpha}}{\beta + r^{\alpha}} dr = \mathbb{E}_r \left[\frac{r^{\alpha}}{\beta + r^{\alpha}} \right].$$

For irregular shapes this can be found through a Monte Carlo simulation.

- When $\mathcal{A}$ is a regular polygon, the solution is given by [1, 4].
- When $\mathcal{A}$ is a disk of radius r_o and M is fixed, then:

$$\epsilon[M] = 1 - e^{-\beta/\mathsf{SNR}} \left(1 + \frac{p\pi}{|\mathcal{A}|} \Phi\left(r_o\right) \right)^{M}$$

where:

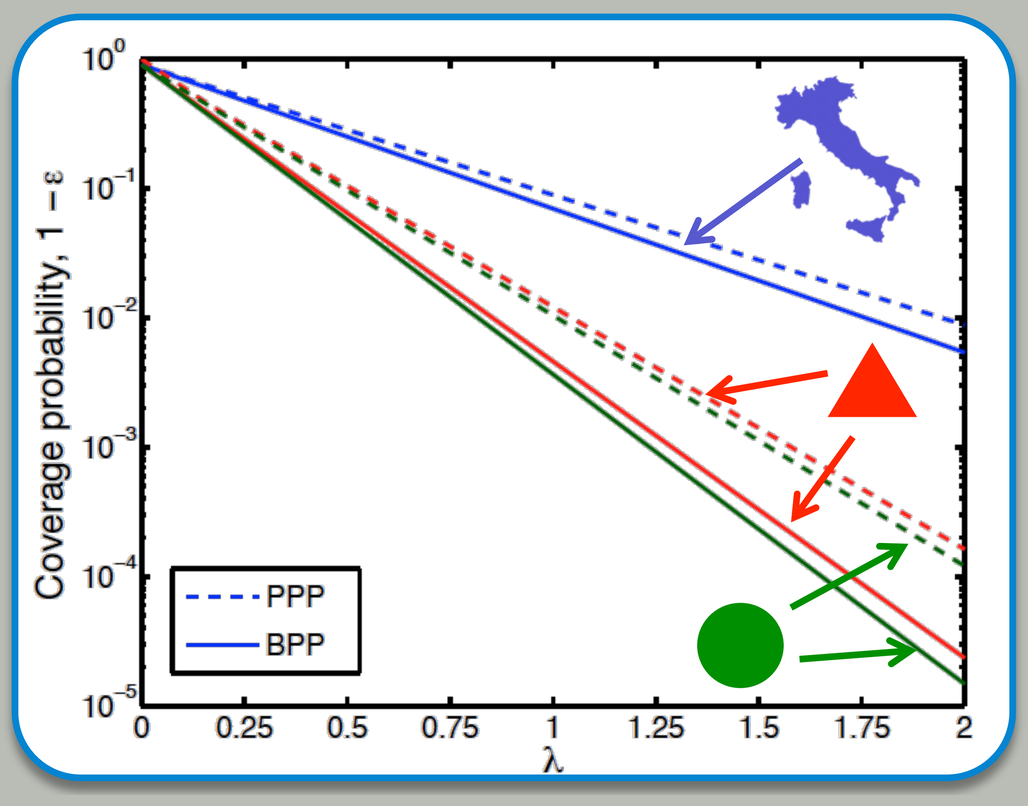
$$\Phi(x) = \frac{2}{\beta} \left(\frac{y^{\alpha+2}}{2+\alpha} \right) {}_2F_1 \left(\left[1, \frac{2}{\alpha} + 1 \right]; \frac{2}{\alpha} + 2; -\frac{y^{\alpha}}{\beta} \right) - x^2.$$

- When M is Poisson (a Poisson point process defined over $\mathcal{A}$), then:

$$\epsilon = 1 - \exp\left\{ -\frac{\beta}{\mathsf{SNR}} + \pi \lambda p \Phi(r_o) \right\} \quad (1)$$

where λ is the intensity of the PPP.

Outage for Finite Regions



The *coverage* probability $(1 - \epsilon)$ for three shapes and the following parameters: $\alpha = 3.2$, $\beta = 0$ dB, SNR $= 10$ dB, $r_o = 2$, and $|\mathcal{A}| = 4\pi$.

Revisiting Baccelli

To obtain the outage probability when the interferers are drawn from a PPP defined over the entire plane, take the limit of (1) as $r_o \to \infty$,

$$\epsilon = 1 - \exp\left\{ -\frac{\beta}{\mathsf{SNR}} + \pi \lambda p \lim_{x \to \infty} \Phi\left(x\right) \right\}. \quad (2)$$

By performing the change of variables $y = \frac{\beta}{x^{\alpha}}$,

$$\lim_{x \to \infty} \Phi\left(x\right) = \lim_{y \to 0} \Phi\left(\frac{\beta^{1/\alpha}}{y^{1/\alpha}} \right) = \frac{2}{\alpha} \beta^{\frac{2}{\alpha}} \lim_{y \to 0} \left\{ y^{-\frac{2}{\alpha}} \phi\left(y\right) \right\} \quad (3)$$

where

$$\phi(y) = \frac{1}{y} \int_0^1 t^{\frac{2}{\alpha}} \left(1 + \frac{t}{y} \right)^{-1} dt - \frac{\alpha}{2}.$$

It is shown in [1] that

$$\lim_{y \to 0} y^{-\frac{2}{\alpha}} \phi(y) = -\sum_{i=-\infty}^{\infty} \frac{(-1)^i}{\frac{2}{\alpha} - i} = -\pi \csc\left(\frac{2\pi}{\alpha} \right). \quad (4)$$

Substituting (4) into (3) and (2) yields

$$\epsilon = 1 - \exp\left\{ -\frac{\beta}{\mathsf{SNR}} - \frac{2\pi^2 \lambda p}{\alpha} \beta^{\frac{2}{\alpha}} \csc\left(\frac{2\pi}{\alpha} \right) \right\} \quad (5)$$

which for SNR $\to \infty$ corresponds to equation (3.4) in [5].

Benefits and Conclusions

- The spatially outage probability can be found without needing to resort to Laplace transforms or the probability generating functional (pgfl).
- The approach is particularly effective for finding outage probability when the network is a BPP or PPP defined over a finite region.
- Through a simple 1-dimensional Monte Carlo simulation, any arbitrary shape can be handled.
- In the limit of a PPP defined over an infinite plane, the result of Baccelli is retrieved.

References

[1] M.C. Valenti, D. Torrieri, and S. Talarico, "A direct approach to computing spatially averaged outage probability," to appear in *IEEE Communications Letters*, 2014.

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[4] J. Guo, S. Durrani, and X. Zhou, "Outage probability in arbitrarily-shaped finite wireless networks," *IEEE Trans. Commun.*, vol. 62, pp. 699–712, Feb. 2014.

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