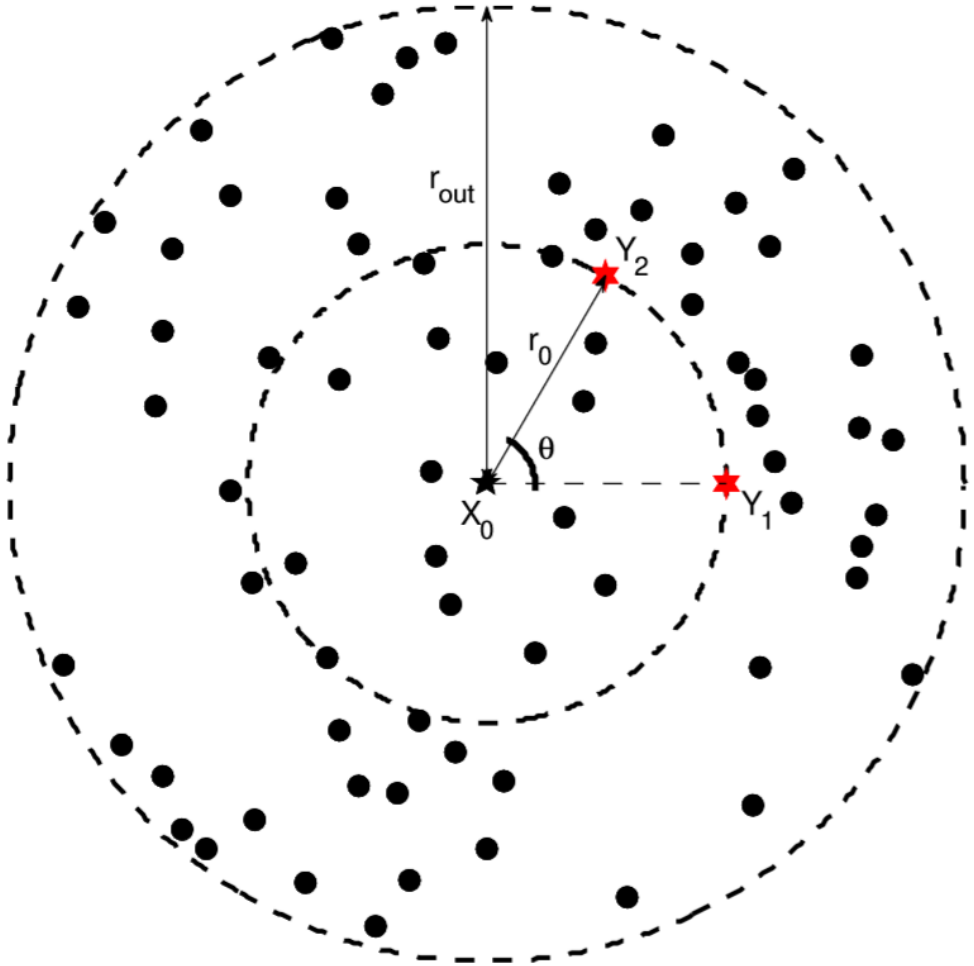
Outage Correlation in Finite and Clustered Wireless Networks

Salvatore Talarico, Intel Mobile Communications Technology
Matthew C. Valenti, West Virginia University
Marco Di Renzo, CNRS-Centrale Supelec-University of Paris-Sud XI

Problem Statement: Quantifying the Outage Correlation



- Suppose you have a **finite network:**
  - A **reference transmitter** is at the center.
  - The network has many **randomly placed interferers**.
  - Signals undergo **Rayleigh fading**.
- A receiver at location **Y_1** will have a certain **outage probability**
  - And so will a receiver at location **Y_2**.
- **Question:** If Y_1 is in an outage, is Y_2 likely to be in an outage?
  - Can be answered by quantifying the **outage correlation**.

Contributions

- The outage probability at a given location is a **random variable.**
  - Randomness is due to **fading** and the interferers' random **location** and **activity.**
- Outage probability at two locations is a **pair** of random variables.
  - Just like any pair of random variables, they can be characterized by a **correlation coefficient**.
  - Main task in this paper is to **compute** the correlation coefficient.
- This problem has been previously solved for **infinite** networks with interferers drawn from a **PPP** [7].
  - Our contribution is to consider more **arbitrary** networks:
    - **Finite** in extent.
    - Interferers drawn from **arbitrary point processes**: BPP, PPP, Clustered PP.
- **Key finding: Interference can be correlated even if the fading is independent.**
  - Reason: Interferers' location and activity are **common sources of randomness**.

[7] R. K. Ganti and M. Haenggi, "Spatial and temporal correlation of the interference in ALOHA ad hoc networks," *IEEE Commun. Letters*, vol. 13, pp. 631–633, September 2009.

SINR

The instantaneous signal-to-interference-and-noise ratio (**SINR**) at the receiver location Y_j is:

Power transmitted by reference transmitter X_0

Path-loss exponent ($\alpha \geq 2$)

$$\gamma_j = \frac{P_0 g_{0,j} r_{0,j}^{-\alpha}}{\mathcal{N} + \sum_{i=1}^{M} I_i P_i g_{i,j} r_{i,j}^{-\alpha}}$$

Number of interferers

Noise Power

Distances between the ith transmitter and the jth receiver

Fading gains (e.g., i.i.d exponential for Rayleigh fading)

Bernoulli random variable indicating interference, where $P[I_i=1]=p_i$)

Outage Probability

- The SINR is a function of:
  - **Fading** (**g**)
  - **Location** of interferers (**r**)
  - **Number** of interferers (M)
- Like any random variable, the SINR can be characterized by its **CDF**:

$$F_{\gamma_j}(\beta) \quad = \quad P\left[\gamma_i \leq \beta\right]$$

- By interpreting β to be an **outage threshold**, then the CDF is the **outage probability** of the SINR

$$\epsilon_j \quad = \quad F_{\gamma_j}(\beta)$$

  - In the above, the dependence on β is suppressed.

From Conditional Outage Probability to Its Spatial Average [5]

Fix r and M, and find the conditional OP averaged over fading:

$$\epsilon_j(M,\boldsymbol{r}) \quad = \quad \mathbb{E}_g[\mathbf{1}\left(\gamma_j \leq \beta\right)|M,\boldsymbol{r}]$$

For a given M, the spatially averaged OP is:

$$\epsilon_j(M) \quad = \quad \mathbb{E}_{\boldsymbol{r}}\left[\epsilon_j\left(\boldsymbol{r}\right)|M\right] = \int \epsilon_j\left(M,\boldsymbol{r}\right) f_{\boldsymbol{r}}\left(\boldsymbol{r}|M\right)\, \mathrm{d}\boldsymbol{r}$$

If interferers are uniform on a circle [24]:

$$f_r(r|M) = \begin{cases} \frac{2\pi r r_0^2}{|\mathcal{A}|} & \text{for } 0 \leq r \leq \tau_1 \\ \frac{2 r r_0^2}{|\mathcal{A}|} \arccos\left(\frac{r_0^2 r^2 + r_0^2 - r_{\text{out}}^2}{2 r_0^2 r}\right) & \text{for } \tau_1 \leq r \leq \tau_2 \end{cases}$$

If M is random, then take the expectation with respect to M:

$$\epsilon_j \quad = \mathbb{E}_M\left[\epsilon_j(M)\right] \quad = \sum_{m=0}^{\infty} p_M[m] \mathbb{E}_{\boldsymbol{r}}\left[\epsilon_i^x\left(\boldsymbol{r}\right)|m\right]$$

$p_M[m]$ is the PMF of M

For PPP:

$$p_M[m] = \frac{(\lambda|\mathcal{A}|)^m}{m!} \exp\left(-\lambda|\mathcal{A}|\right), \quad \text{for } m \geq 0$$

[5] M. C. Valenti, D. Torrieri, and S. Talarico, “A direct approach to computing spatially averaged outage probability,” *IEEE Commun. Letters*, vol. 18, pp. 1103–1106, July 2014.

[24] Z. Khalid and S. Durrani, “Distance distributions in regular polygons,” *IEEE Trans. Veh. Tech.*, vol. 62, pp. 2363–2368, June 2013.

Averaging Over the Fading

- When **conditioned** on the location and number of interferers, the **outage probability** is:

$$\epsilon_j(M,\boldsymbol{r}) = \mathbb{E}_g[\mathbf{1}(\gamma_j \le \beta)\,|M,\boldsymbol{r}]$$

$$= 1-\exp\left(-\frac{\beta}{\mathsf{SNR}}\right)\prod_{i=1}^{M}\left(1-p_i+\frac{p_i d_{i,j}^{\alpha}}{\beta+d_{i,j}^{\alpha}}\right)$$

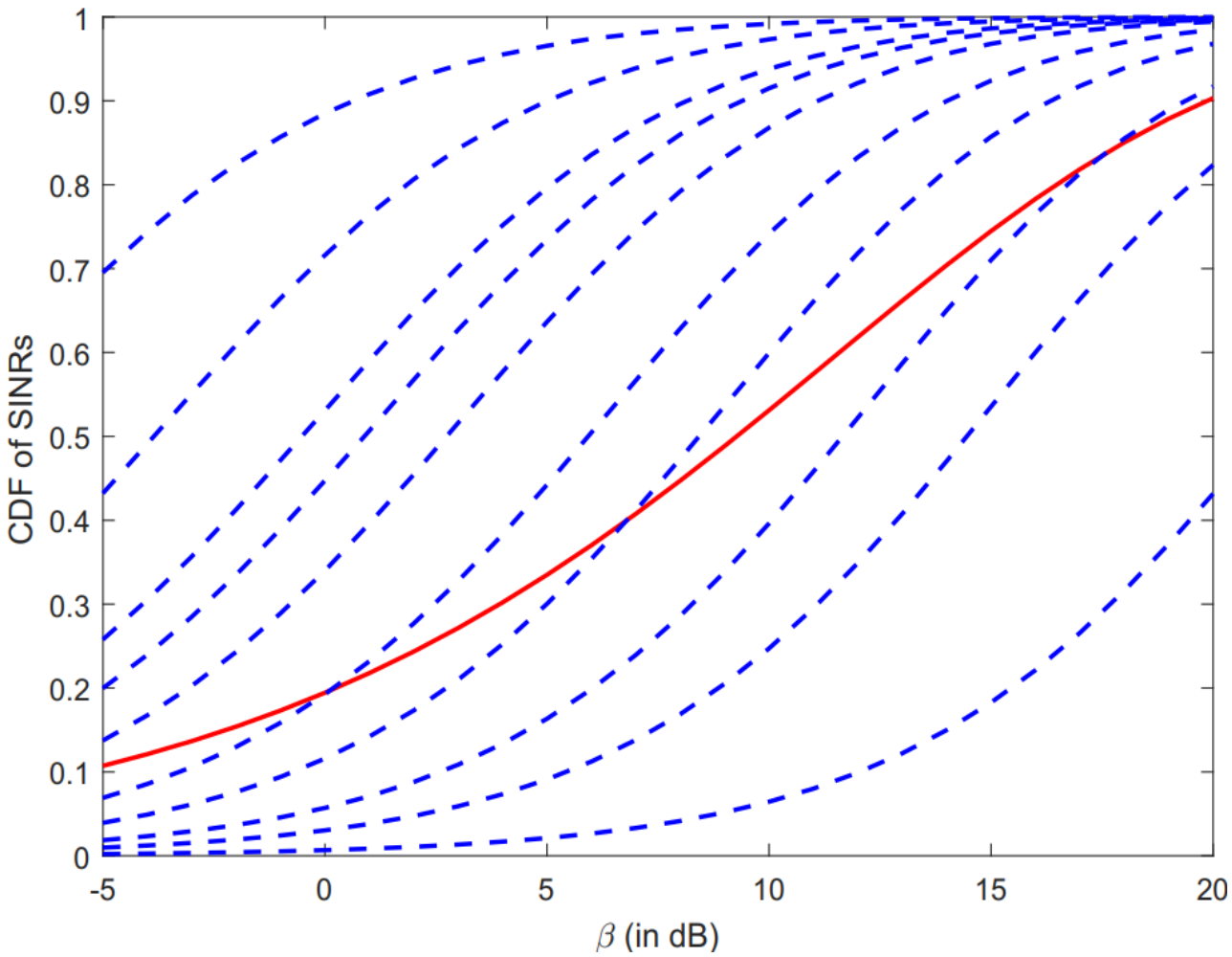
Expectation operator evaluated over the fading

Indicator function

$$\mathsf{SNR} = P_0 r_0^{-\alpha}/\mathcal{N} \qquad d_{i,j} = r_{i,j}/r_0$$

- The set $\{\epsilon_1(M,\mathbf{r}),\epsilon_2(M,\mathbf{r})\}$ constitutes a **pair of random variables**.
  - Despite the independent fading, the two variables may be **correlated** because they both depend on the same realization of interference locations (**r**)

Outage Probability and Its Dependence Upon Topology



Fading	Rayleigh
Network area	$\pi \bullet r_{out}^2$ with $r_{out}=1$
Distance between X_0 and Y_j	$r_0=0.25$
Path-loss exponent	$\alpha=3.5$
SNR	10 dB
Number of Interferers	M=2

- Blue dashed lines = conditional outage probability
- Red = spatial average
- The outage probability **varies significantly** with the **location of the interferers**.

Outage Correlation Coefficient

- The **correlation coefficient** of the outage at the two receivers is:

$$\zeta[Y_1, Y_2] = \frac{\mathbb{E}_{M,\mathbf{r}}\left[\epsilon_1(M,\mathbf{r})\,\epsilon_2(M,\mathbf{r})\right] - \mathbb{E}_{M,\mathbf{r}}\left[\epsilon_1(M,\mathbf{r})\right]\mathbb{E}_{M,\mathbf{r}}\left[\epsilon_2(M,\mathbf{r})\right]}{\sqrt{\mathbb{E}_{M,\mathbf{r}}\left[\epsilon_1^2(M,\mathbf{r})\right] - \mathbb{E}_{M,\mathbf{r}}^2\left[\epsilon_1(M,\mathbf{r})\right]}\sqrt{\mathbb{E}_{M,\mathbf{r}}\left[\epsilon_2^2(M,\mathbf{r})\right] - \mathbb{E}_{M,\mathbf{r}}^2\left[\epsilon_2(M,\mathbf{r})\right]}}$$

- The above **expectation** is with respect to r and M.
- Requires computation of:
  - **First moments** (same procedure as for finding spatial average outage probability)
  - **Second** moments
  - **Joint** first moment
- If M is fixed (e.g., a BPP), and if the two SINRs are stochastically equivalent (i.e., $|Y_1| = |Y_2| = r_0$), then the interference correlation coefficient can be written as:

$$\zeta[Y_1, Y_2] = \frac{\mathbb{E}_{\mathbf{r}}\left[\epsilon_1(\mathbf{r})\,\epsilon_2(\mathbf{r})\right] - \mathbb{E}_{\mathbf{r}}^2\left[\epsilon_j(\mathbf{r})\right]}{\mathbb{E}_{\mathbf{r}}\left[\epsilon_j^2(\mathbf{r})\right] - \mathbb{E}_{\mathbf{r}}^2\left[\epsilon_j(\mathbf{r})\right]}$$

Approach to Evaluate Correlation Coefficient

Fix r and M, and find the conditional OP averaged over fading:

$$\epsilon_j(M,\boldsymbol{r}) \quad = \quad \mathbb{E}_g[\mathbf{1}\left(\gamma_j \leq \beta\right)|M,\boldsymbol{r}]$$

For a given M, evaluate the spatial average:

- The spatially averaged OP is:

$$\mathbb{E}_{\boldsymbol{r}}\left[\epsilon_j\left(\boldsymbol{r}\right)|M\right] = \int \epsilon_j\left(M,\boldsymbol{r}\right) f_{\boldsymbol{r}}\left(\boldsymbol{r}|M\right)\,\mathrm{d}\boldsymbol{r}$$

- The spatially averaged second moment OP is:

$$\mathbb{E}_{\boldsymbol{r}}\left[\epsilon_j^2\left(\boldsymbol{r}\right)|M\right] = \int \epsilon_j^2\left(M,\boldsymbol{r}\right) f_{\boldsymbol{r}}\left(\boldsymbol{r}|M\right)\,\mathrm{d}\boldsymbol{r}$$

- The spatially averaged first joint moment is:

$$\mathbb{E}_{\boldsymbol{r}}\left[\epsilon_1\left(\boldsymbol{r}\right)\epsilon_2\left(\boldsymbol{r}\right)|M\right] = \int\int \epsilon_1(M,l\left(\boldsymbol{\rho},\boldsymbol{\phi},0\right))\epsilon_2(M,l\left(\boldsymbol{\rho},\boldsymbol{\phi},\theta\right))f_{\rho}\left(\boldsymbol{\rho}\right)f_{\phi}\left(\boldsymbol{\phi}\right)\mathrm{d}\boldsymbol{\rho}\,\mathrm{d}\boldsymbol{\phi}$$

For BPP:

$$f_r(r|M)=\begin{cases}\frac{2\pi r r_0^2}{|\mathcal{A}|} & \text{for } 0\leq r\leq \tau_1\\ \frac{2rr_0^2}{|\mathcal{A}|}\arccos\left(\frac{r_0^2r^2+r_0^2-r_{\text{out}}^2}{2r_0^2r}\right) & \text{for } \tau_1\leq r\leq \tau_2\end{cases}$$

$$l\left(\boldsymbol{\rho},\boldsymbol{\phi},\theta\right)=||r_0\exp\left(j\theta\right)+r_{\text{out}}\sqrt{\boldsymbol{\rho}}\exp\left(j\boldsymbol{\phi}\right)||$$

$$\rho \sim \mathcal{U}\left(0,1\right)$$

$$\phi \sim \mathcal{U}\left(0,2\pi\right)$$

If M is random, then take the expectation with respect to M:

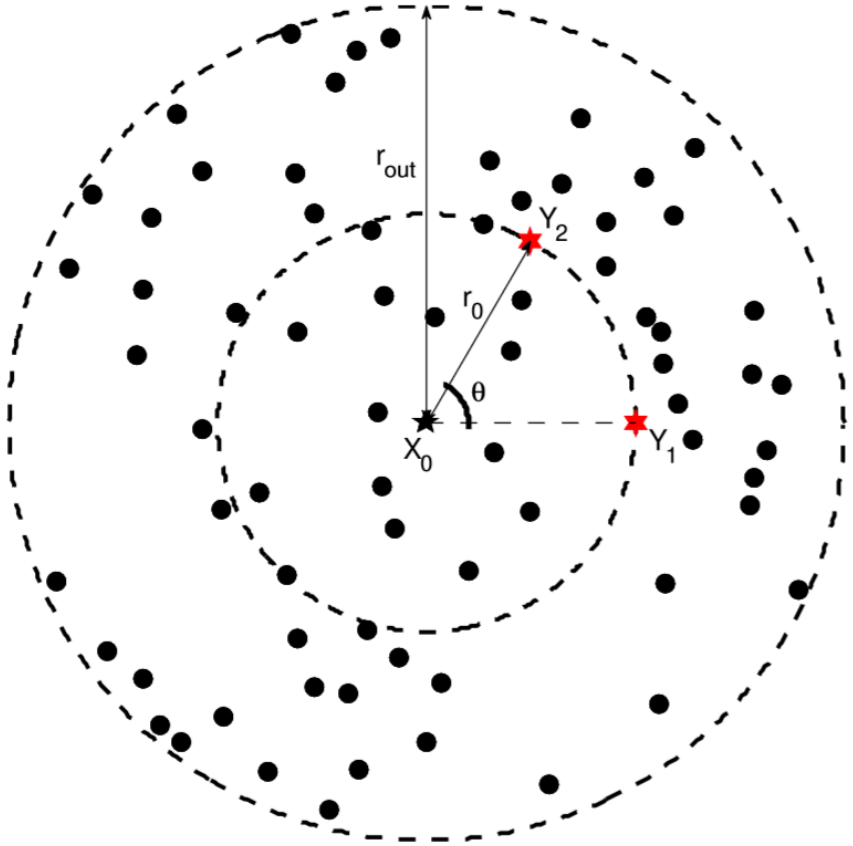
$$\mathbb{E}_{M,\boldsymbol{r}}\left[\epsilon_i^x\left(\boldsymbol{r}\right)\epsilon_j^y\left(\boldsymbol{r}\right)\right]=\sum_{m=0}^{\infty}p_M[m]\mathbb{E}_{\boldsymbol{r}}\left[\epsilon_i^x\left(\boldsymbol{r}\right)\epsilon_j^y\left(\boldsymbol{r}\right)|m\right]$$

$p_M[m]$ is the PMF of M

For PPP:

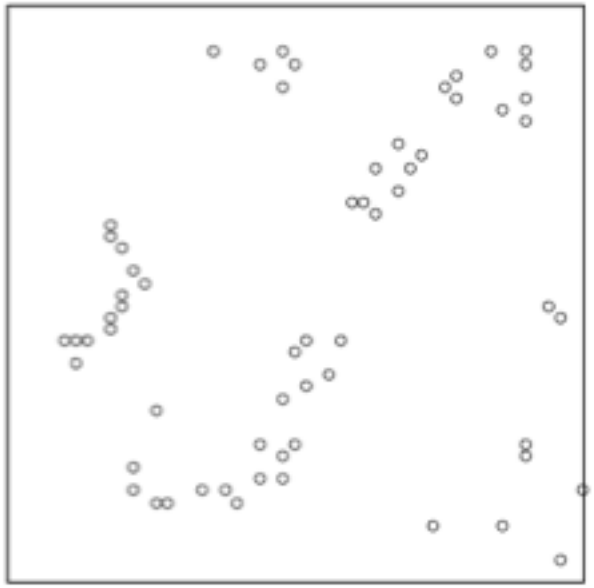
$$p_M[m]=\frac{(\lambda|\mathcal{A}|)^m}{m!}\exp\left(-\lambda|\mathcal{A}|\right),\quad \text{for } m\geq 0$$

Numerical Results

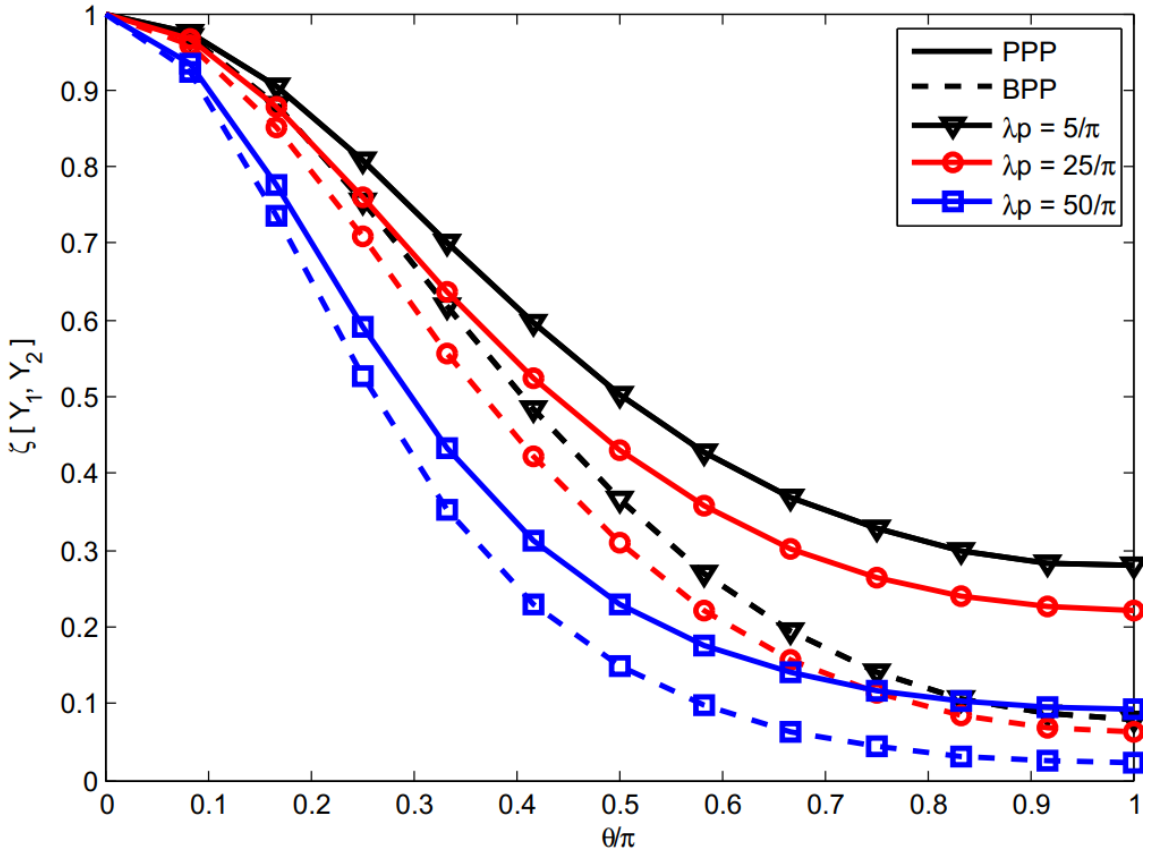


- Analytical results are verified via simulation.
- Three point processes are investigated:
  - Binomial Point Process (BPP) w/ M interferers.
  - Poisson Point Process (PPP) w/ intensity λ
  - Thomas Cluster Process (TCP)

Fading	Rayleigh
Radius of network	$r_{out}=1$
Distance between X_0 and Y_j	$r_0=0.25$
Path-loss exponent	$\alpha=3.5$
SNR	10 dB
SINR threshold	$\beta=0$ dB



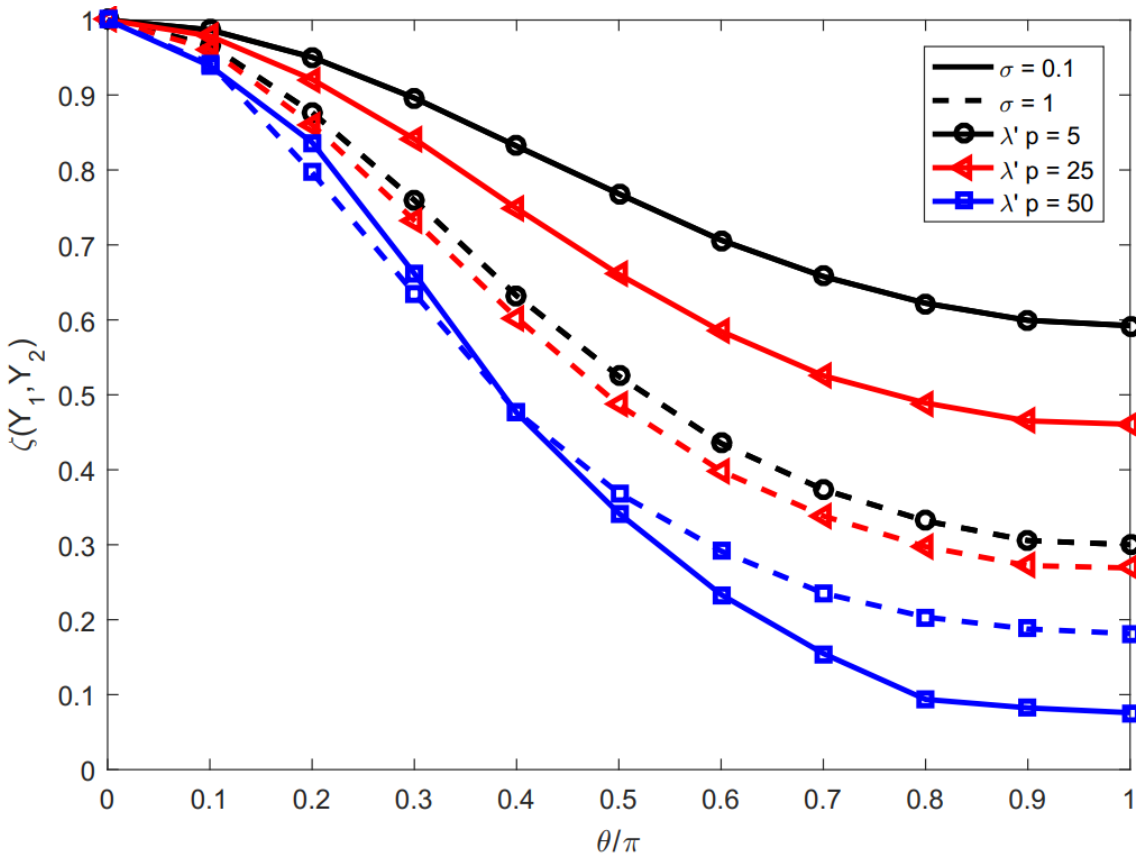
Results for BPP & PPP



Fading	Rayleigh
Network area	$\pi \bullet r_{out}^2$ with $r_{out}=1$
Distance between X_0 and Y_j	$r_0=0.25$
Path-loss exponent	$\alpha=3.5$
SNR	10 dB
SINR threshold	$\beta=0$ dB

- Sparser networks are more correlated than denser networks (denser network corresponds to higher $\lambda p = \frac{\mathbb{E}[M]p}{|\mathcal{A}|}$).
- A PPP experiences a higher correlation than a BPP.

Results for Thomas Cluster Process



Fading	Rayleigh
Network area	$\pi \bullet r_{out}^2$ with $r_{out}=1$
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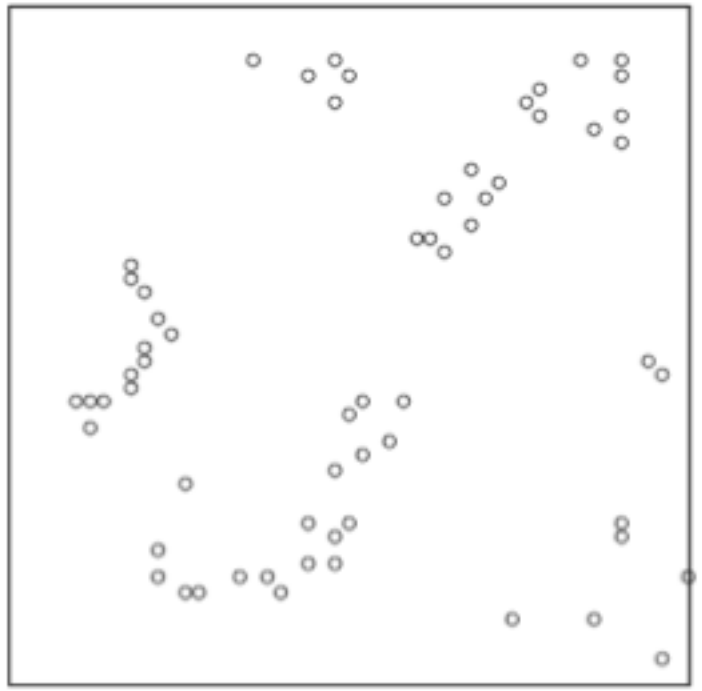

- A more compact cluster (a cluster characterized by a lower σ) experiences higher correlation.
- As the network becomes denser the correlation decreases faster, since there are more dominant interferers.

Conclusions

- Summary:
  - An analytical framework to **evaluate** in closed-form the **correlation coefficient** of the outage probability of a finite wireless network is provided.
  - The proposed approach is used to evaluate analytical expressions for **several point processes** (i.e., BPP, PPP, TCP).
- Remarks:
  - **Sparser** networks have a **higher** spatial correlation than denser networks.
  - PPP networks have a **higher** correlation than BPP.
  - The spatial correlation in a TCP is **smaller** when the offspring points are **more highly dispersed**.
- Applications:
  - Cooperative protocols
  - Caching
  - User-centric communications