COMBINED MULTIUSER DETECTION AND CHANNEL DECODING WITH RECEIVER DIVERSITY *

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Abstract

A reception strategy involving the concepts of multiuser detection, macrodiversity, and iterative decoding is applied to the uplink of a convolutionally coded time division multiple-access (TDMA) cellular network composed of K asynchronous cochannel transmitters (mobiles) and M geographically separated receivers (base stations). At each receiver, a soft-output multiuser detector (MUD) is implemented using the Log-MAP algorithm. The outputs of the M base stations are combined and passed to a bank of K soft-output convolutional decoders, also implemented using the Log-MAP algorithm. The outputs of the decoders are fed back to the base stations to be used as a priori information during the next iteration of MUD. Processing continues in an iterative manner analogous to the decoding of turbo codes. Simulation results are shown for the first tier of cochannel interferers in a fully-interleaved Rayleigh flat-fading channel with 120 degree sectorized antennas at the base stations. The results indicate significant performance gains in the presence of severe cochannel interference, and suggest a possible reduction in the frequency reuse distance.

I. Introduction

The performance of the uplink in a digital cellular communication system is primarily limited by multiple-access interference and multipath fading. The detrimental effects of multiple-access interference can be alleviated by the use of multiuser detection, which in its optimal form is implemented by the Viterbi algorithm of 2^K states, where K is the number of cochannel transmitters (users) [1]. In addition, the effects of both multiple-access interference and multipath fading can be mitigated by the use of powerful forward error correction (FEC) coding and interleaving. Thus a receiver structure that incorporates both multiuser detection and FEC decoding can be a powerful countermeasure to the hostile environment typical of many mobile communication channels.

In [2], a receiver structure for the decoding of convolutionally coded asynchronous multiple-access signals is derived. This receiver performs maximum likelihood sequence detection (using the Viterbi algorithm) over the super-trellis created by the combination of the trellis of the convolutional code and the trellis due to multiple-access interference. Although optimal (in terms of asymptotic multiuser coding gain), this approach suffers from two problems. First, the complexity per bit of this approach is $O(2^{WK})$, where W is the code constraint length, and thus is not feasible even for short constraint lengths and a small number of users. Second, the super-trellis does not account for interleaving and thus this approach cannot be used if the code symbols are interleaved prior to transmission.

An alternative approach to the joint multiuser detection and FEC decoding problem has been inspired by recent research in the area of turbo codes [3]. The problem of detecting turbo codes is analogous to the problem of joint multiuser detection and FEC decoding. In both cases, the receiver is supplied with observations of two Markov processes that are driven indirectly by the same data. With turbo codes, the two Markov processes are parallel concatenated recursive systematic convolutional (RSC) codes separated by an interleaver. Likewise, the problem of joint multiuser detection and FEC decoding can be decomposed into two Markov processes, the multiple-access channel and the FEC code, which are serially concatenated and possibly separated by an interleaver. Thus the same decoder structure that is used for turbo codes can be used to perform joint multiuser detection and FEC decoding. As suggested

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in [4], we prefer to use the term "turbo-processing" for the generalization of the turbo decoding algorithm.

In order to apply turbo-processing to the joint MUD/FEC-decoding problem, both the MUD and the FEC decoder must be implemented with soft-input soft-output (SISO) trellis-based decoding algorithms. The maximum a posteriori symbol-by-symbol estimation (MAP) algorithm and the soft-output Viterbi algorithm (SOVA) have been identified as candidates for turbo processing. Turbo-processing is an iterative procedure. During the first iteration, decision statistics from a matched filter are passed to the multiuser detector. The output of the MUD is then passed to an FEC channel decoder. Next, the output of the FEC channel decoder is fed back to the multiuser detector to be used as a priori information during the next iteration. The turbo-processing loop is iterated several times until the desired performance is attained, although the incremental benefit of performing additional iterations obeys a law of diminishing returns.

The application of turbo-processing to the joint MUD/FEC-decoding problem was applied to synchronous direct-sequence code division multiple-access (DS-CDMA) networks in [5], to asynchronous DS-CDMA networks in [6], and to time division multiple-access (TDMA) networks in [7]. Additionally, in [8], it was shown that turbo-processing could be used for synchronous systems with a high degree of signal correlation, such as frequency division multiple-access (FDMA) with significant frequency overlap. In all of these examples, the coded performance of a heavily loaded system approaches that of a single-user system after just a few iterations of turbo-processing.

The results shown in [7] and [8] indicate that the use of turbo-processing can greatly improve the performance of systems with highly correlated signals, such as TDMA systems with a low frequency reuse distance. However, before discussing the application of turbo-processing to TDMA systems it is important to first note how the multiuser detection problem for TDMA differs from that for DS-CDMA. When used for DS-CDMA, a single multiuser detector jointly detects all K signals that originate from within a single cell (intracell interference). However, when used for TDMA, each multiuser detector jointly detects K cochannel signals originating from a set of M cells (intercell interference). Because the cochannel signals in a TDMA system originate from separate cells, and because each cell has its own serving base station, it can be assumed that each base station has its own multiuser detector. Rather than just using the output of any one multiuser detector as the input to the FEC decoders, we suggest that the output of all M multiuser detectors be used as input to the FEC decoders. Because the output of each multiuser detector is in the form of a log-likelihood ratio, the outputs of the multiuser detectors can simply be added together to form the input to the FEC decoders. The general strategy of combining outputs from different base stations is referred to as "macrodiversity" or "base station diversity", and its application to cellular systems is discussed in [9]. In [10], the concept of macrodiversity is extended to uncoded TDMA systems that incorporate multiuser detection at each base station.

In this paper, we discuss a new reception technique for TDMA systems composed of K mobiles and M base stations. Each base station consists of a soft-input soft-output multiuser detector implemented with the log-MAP algorithm of [11]. The outputs of the M base stations, which are in the form of log-likelihood ratios, are added together and passed to a bank of K channel decoders, also implemented with the log-MAP algorithm. The output of the bank of channel decoders is then fed back to the base station to be used as a priori information during the next iteration of turbo-processing. Several iterations of this processing loop are executed before the channel decoders produce their final estimate. In section II, the system model is presented. In section III, the details of the reception technique are discussed. Section IV presents a simulation example for a TDMA network comprised of $K = 3$ mobiles and $M = 3$ base stations which models the effect of the first tier of cochannel interference when 120 degree sectorized antennas are used at the base stations. A fully-interleaved Rayleigh flat-fading channel is assumed, and a simple constraint length $W = 3$ rate $R = 1/2$ convolutional code is employed. Finally conclusions are drawn in section V.

II. System Model

Consider a TDMA network consisting of K asynchronous cochannel binary phase-shift keying (BPSK) transmitters and M spatially separated receivers. At transmitter k, a bit sequence, $\mathbf{d}_k = \{d_k[j]\}$ for $1 \leq j \leq L/n$, the last m_c of which are zeros, is passed through a rate $R = 1/n$ convolutional encoder with constraint length $W = m_c + 1$. The frame of L code symbols is BPSK modulated and sent over a flat-fading multiple-access channel. The signal at receiver m is the noisy sum of K delayed and attenuated signals contributed by each of the K transmitters:

$$r_m(t) = \sum_{k=1}^{K} s_{k,m}(t) + n_m(t), \qquad (1)$$

where $n_m(t)$ is a complex additive white Gaussian noise (AWGN) process with power spectral density

$N_o/2$ and $s_{k,m}(t)$ is the signal at receiver m due to user k:

$$s_{k,m}(t) = \sum_{i=1}^{L} \sqrt{P_{k,m}[i]} b_k[i] \Pi(t - iT - \tau_{k,m}) e^{j\phi_{k,m}}. \quad (2)$$

$P_{k,m}[i]$ is the power of user k at receiver m during the ith signaling interval, $b_k[i] \in \{-1, 1\}$ is the ith coded symbol of user k, $\tau_{k,m}$ is the delay of user k at receiver m (assumed to be uniformly distributed over one symbol period T), and $\phi_{k,m}$ is the carrier phase of user k at receiver m (assumed to be uniformly distributed from 0 to 2π radians). It is assumed that the system does not use pulse shaping, and thus $\Pi(t)$ is the rectangular pulse function defined by:

$$\Pi(t) = \begin{cases} 1, & \text{for } 0 \le t < T \\ 0, & \text{elsewhere.} \end{cases} \quad (3)$$

The indices of the K users are assigned such that the relative delays $\tau_{k,m}$ are in ascending order, i.e. $\tau_{1,m} \le \tau_{2,m} \le ... \le \tau_{K,m}$.

For the AWGN channel, the power $P_{k,m}[i]$ is constant for all i, while for the fully-interleaved Rayleigh flat-fading channel, each amplitude $\sqrt{P_{k,m}[i]}$ is an independent realization of a Rayleigh random variable. Fading is assumed to be slow, and thus the fading amplitude is constant for the entire symbol interval.

The received signal is passed through a bank of matched filters with output:

$$y_{k,m}[i] = \int_{-\infty}^{\infty} \Re\left\{ r_m(t) \Pi(t - iT - \tau_{k,m}) e^{-j\phi_{k,m}} \right\} dt, \quad (4)$$

where $y_{k,m}[i]$ denotes the matched filter output at receiver m corresponding to the ith symbol of user k, and $\Re\{\cdot\}$ denotes the real part. The vector $\mathbf{y}_m$ of matched filter outputs constitutes a set of sufficient statistics for receiver m and is created according to:

$$\mathbf{y}_m = [y_{1,m}[1], ..., y_{K,m}[1], ..., y_{1,m}[L], ..., y_{K,m}[L]]^T. \quad (5)$$

The set of matched filter outputs from the M independent receivers are collected into a matrix $\mathbf{Y}$:

$$\mathbf{Y} = [\mathbf{y}_1, ..., \mathbf{y}_m, ..., \mathbf{y}_M]. \quad (6)$$

Similarly, a vector of symbols is defined by:

$$\mathbf{b} = [b_1[1], ..., b_K[1], ..., b_1[L], ..., b_K[L]]^T. \quad (7)$$

A vector of fading amplitudes $\mathbf{c}_m$ at receiver m is created according to:

$$\mathbf{c}_m = \left[\sqrt{P_{1,m}[1]}, ..., \sqrt{P_{K,m}[1]}, ..., \sqrt{P_{1,m}[L]}, ..., \sqrt{P_{K,m}[L]} \right]^T. \quad (8)$$

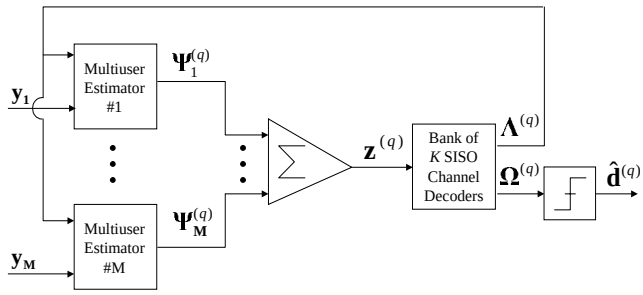

Figure 1: Proposed system architecture.

The fading amplitudes at all M receivers are collected into the matrix $\mathbf{C}$:

$$\mathbf{C} = [\mathbf{c}_1, ..., \mathbf{c}_m, ..., \mathbf{c}_M]. \quad (9)$$

Note that the column vectors $\mathbf{y}_m$, $\mathbf{b}$, and $\mathbf{c}_m$ have $N = LK$ elements, and the matrices $\mathbf{Y}$ and $\mathbf{C}$ are $N \times M$.

III. Distributed Coded Multiuser Reception

The proposed turbo processing architecture is shown in Figure 1. At each base station, the matched filter output $\mathbf{y}_m$ is fed into a soft-input soft-output (SISO) multiuser detector (also called a multiuser estimator). The log-likelihood ratios $\left\{\mathbf{\Psi}_m^{(q)}\right\}$ that are output by the M multiuser estimators during iteration q are added together and sent to a bank of K SISO channel decoders. The output of the bank of decoders $\mathbf{\Lambda}^{(q)}$ is sent back to the multiuser estimators to be used as a priori information for iteration $q+1$. The hard system output for the qth iteration, $\hat{\mathbf{d}}_k^{(q)}$, is found by limiting the output of the channel decoder.

A. Multiuser Detection

The optimal multiuser detector at receiver m for iteration q selects the vector of estimated transmitted bits according to [1]:

$$\hat{\mathbf{b}}_m^{(q)} = \arg\left\{ \max_{\mathbf{b}} \sum_{i=1}^{N} \lambda_{i,m}(\mathbf{b}) \right\}, \quad (10)$$

where:

$$\lambda_{i,m}(\mathbf{b}) = \frac{n}{E_b/N_o} \ln P\left[b_i | \mathbf{\Lambda}^{(q-1)} \right] + b_i c_{i,m} \left[2y_{i,m} - b_i c_{i,m} - 2 \sum_{d=1}^{K-1} b_{i-d} c_{i-d,m} G_{K-d,\kappa(i)}^{(m)} \right], \quad (11)$$

where $b_i = 0$ for $i \leq 0$ and $\kappa(i) = K$ if K divides i, otherwise $\kappa(i)$ is i modulo K. $\mathbf{G}^{(m)}$ is a matrix of signal cross-correlations at receiver m:

$$G_{i,j}^{(m)} = \frac{1}{T}\cos\left(\phi_{i+j} - \phi_j\right)\left(\tau_{i+j,m} - \tau_{j,m}\right),$$

when $i + j \leq K$, and:

$$G_{i,j}^{(m)} = \frac{1}{T}\cos\left(\phi_{i+j-K} - \phi_j\right)\left[1 - \left(\tau_{j,m} - \tau_{i+j-K,m}\right)\right],$$

when $i + j > K$. The entries of the jth column of $\mathbf{G}^{(\mathbf{m})}$ are the correlations at receiver m of the jth user with the $K-1$ preceeding users.

$P\left[b_i|\mathbf{\Lambda}^{(q-1)}\right]$ is related to the soft output of the channel decoder during the $(q-1)^{th}$ iteration by:

$$P\left[b_i|\mathbf{\Lambda}^{(q-1)}\right] = \begin{cases} \frac{\exp\left\{\Lambda_i^{(q-1)}\right\}}{1+\exp\left\{\Lambda_i^{(q-1)}\right\}}, & \text{for } b_i = +1 \\ \frac{1}{1+\exp\left\{\Lambda_i^{(q-1)}\right\}}, & \text{for } b_i = -1. \end{cases} \tag{12}$$

During initialization, $\mathbf{\Lambda}^{(0)}$ is set to be all zeros.

B. Multiuser Estimation

The estimate $\hat{\mathbf{b}}_m$ can be recursively computed using the Viterbi algorithm with the branch metric in (11). However, use of the standard Viterbi algorithm results in hard bit decisions, which are not suitable for turbo processing. Instead, soft outputs in the form of log-likelihood ratios (LLR) are desired from the multiuser receiver:

$$\Psi_{i,m}^{(q)} = \log\frac{P\left[\hat{b}_{i,m}^{(q)} = +1|\mathbf{y}_m, \mathbf{\Lambda}^{(q-1)}\right]}{P\left[\hat{b}_{i,m}^{(q)} = -1|\mathbf{y}_m, \mathbf{\Lambda}^{(q-1)}\right]}. \tag{13}$$

The above LLR can be calculated using the maximum a posteriori symbol-by-symbol estimation algorithm [12]. However, this algorithm suffers from a high computation complexity and numerical instability. These problems are alleviated by performing the computations in the log domain using the Log-MAP algorithm of [11].

C. Macrodiversity Combining

When soft-output multiuser detection is used at each base station, performance can be improved further by using macrodiversity combining. Since the outputs of the M multiuser detectors are each in the form of a log-likelihood ratio, an equal gain combining rule is optimal:

$$z_i^{(q)} = \sum_{m=1}^{M} \Psi_{i,m}^{(q)}. \tag{14}$$

Note that maximum ratio combining is not necessary here because the branch metric already incorporates the signal amplitudes.

D. Channel Decoding

The serial output, $\mathbf{z}^{(q)} = \{z_i^{(q)}\}$ for $1 \leq i \leq N$, of the adder is split into K parallel sequences $\{\mathbf{z}_k^{(q)}\}$, where $\mathbf{z}_k^{(q)} = \{z_{(i-1)K+k}^{(q)}\}$ for $1 \leq i \leq L$. The decision statistic $\mathbf{z}_k^{(q)}$ for user k is fed into a soft-output channel decoder (implemented with the Log-MAP algorithm), which produces two output sequences. The first output is the log-likelihood ratio of the kth user's FEC encoder *output*:

$$\Lambda_k^{(q)}[i] = \log\frac{P\left[\hat{b}_k^{(q)}[i] = +1|\mathbf{z}_k^{(q)}\right]}{P\left[\hat{b}_k^{(q)}[i] = -1|\mathbf{z}_k^{(q)}\right]}, \tag{15}$$

which are gathered into a vector according:

$$\mathbf{\Lambda}^{(q)} = \left[\Lambda_1^{(q)}[1], ..., \Lambda_K^{(q)}[1], ..., \Lambda_1^{(q)}[L], ..., \Lambda_K^{(q)}[L]\right]^T. \tag{16}$$

This vector is fed back to all M base stations and is used as a priori information during the $(q+1)^{th}$ iteration of turbo-processing.

The second output of the channel decoder for user k is the log-likelihood ratio of the channel encoder *input*:

$$\Omega_k^{(q)}[j] = \log\frac{P\left[\hat{d}_k^{(q)}[j] = +1|\mathbf{z}_k^{(q)}\right]}{P\left[\hat{d}_k^{(q)}[j] = -1|\mathbf{z}_k^{(q)}\right]} \tag{17}$$

The system's hard data bit decisions for iteration q are found by limiting this output:

$$\hat{d}_k^{(q)}[j] = \begin{cases} +1, & \text{if } \Omega_k^{(q)}[j] \geq 0 \\ -1, & \text{if } \Omega_k^{(q)}[j] < 0 \end{cases} \tag{18}$$

IV. Simulation Results

To illustrate the potential gains of the proposed reception technique, the uplink performance of a TDMA cellular network composed of $K = 3$ mobiles and $M = 3$ base stations was simulated. When 120 degree sectorized antennas are used in the base stations there are two cochannel interferers in the first tier. By using a multiuser detector capable of jointly detecting $K = 3$ signals at each base station, the desired signal and first tier of interferers are jointly detected. A simple illustration of the network topology is shown in Figure 2, and it is assumed that each of the $M = 3$ serving base stations contains a multiuser estimator. For the simulation, a rate $R = 1/2$ convolutional code with constraint length $W = 3$ and generator $(5, 7)$ was used. The coded frame size is $L = 4,096$ symbols, and a 64×64 block interleaver is used to interleave symbols

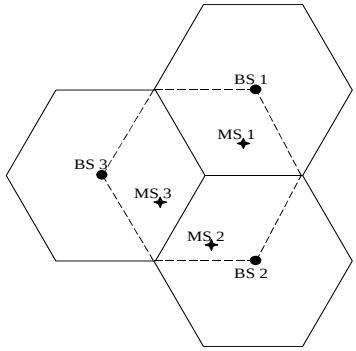

Figure 2: Simulated network topology.

prior to transmission. All multiuser estimators and FEC decoders are implemented with the Log-MAP algorithm. The channel is fully-interleaved Rayleigh flat-faded.

The bit error rate (BER) performance of this system is a function of two parameters: (1) the E_b/N_o ratio, where E_b is the energy per bit and N_o is the single-sided noise spectral density, and (2) the carrier to interference ratio C/I, which we define to be the ratio of the average power of the signal closest to the base station divided by the average power of each of the $K-1$ interferers (it is assumed that all interferers have equal power). Figure 3 shows coded performance as a function of E_b/N_o for a C/I ratio of 7 dB, which is the maximum C/I that can be expected in a modern digital TDMA system such as GSM or IS-136. Figure 4 shows coded performance as a function of C/I when $E_b/N_o = 6$ dB, which was chosen because it was the largest value of E_b/N_o that we could simulate the system with our current computational resources.

In each figure, the performance of three reception techniques are compared. Conventional reception exhibits the worst BER performance and consists of simply decoding the matched filter output of the base station closest to the desired user (i.e. neither multi-user detection or macrodiversity are used). Performance can be improved by maximal-ratio combining the outputs fo the conventional receivers at the M base stations (this method is denoted "macrodiversity using conventional receivers"). Finally, the proposed technique of joint MUD/FEC-decoding with macrodiversity is shown for the first four iterations of turbo-processing. We also simulated a fifth iteration, but the incremental gain over the fourth iteration was negligible.

In Figure 3, a BER floor appears for the conventional receiver. While using macrodiversity alone can lower this floor, it cannot make the floor disappear. But by using the proposed turbo-processing technique the floor appears to disappear after just the first iteration. After the fourth iteration of decoding,

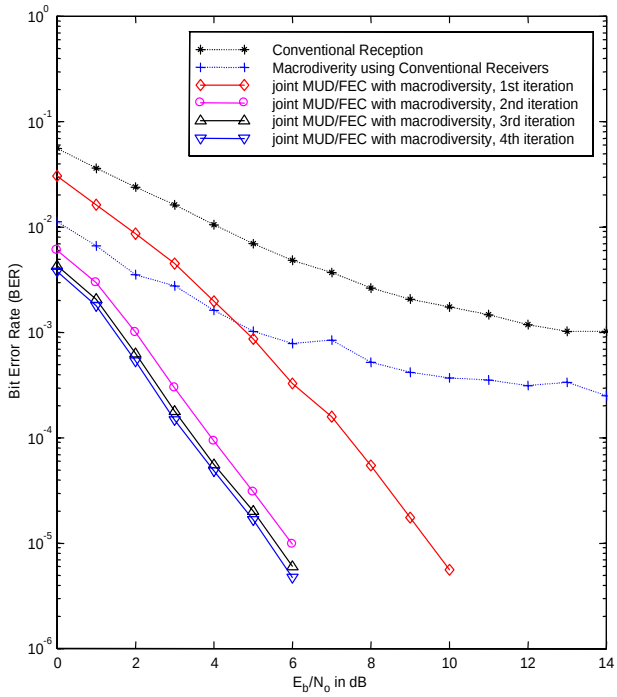

Figure 3: Coded bit error performance of a $K = 3$ transmitter, $M = 3$ receiver TDMA system in a fully-interleaved Rayleigh flat-fading channel as a function of E_b/N_o for various reception and diversity techniques. The carrier-to-interference ratio $C/I = 7$ dB.

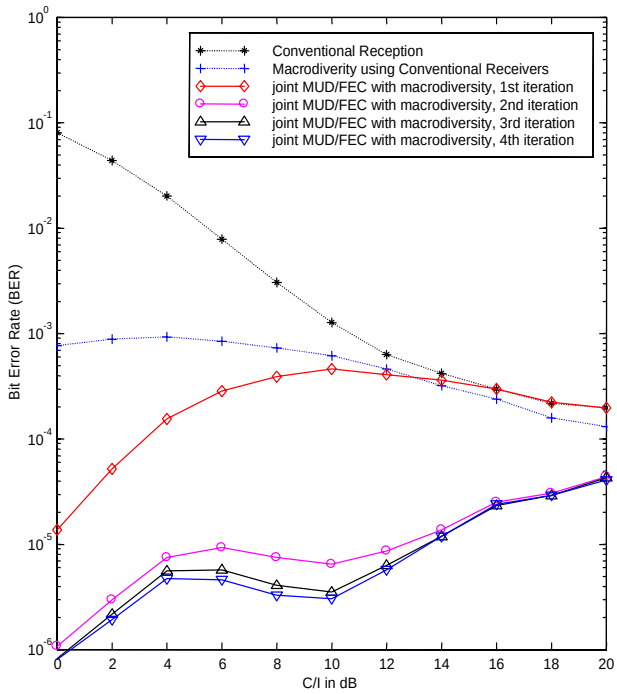

Figure 4: Coded bit error performance of a $K = 3$ transmitter, $M = 3$ receiver TDMA system in a fully-interleaved Rayleigh flat-fading channel as a function of carrier to noise ratio C/I for various reception and diversity techniques. $E_b/N_o = 6$ dB.

the performance is 4 dB superior to the performance of the first iteration. It is interesting to note the significant gain from the first to the second iteration of turbo-decoding. This is because during the first iteration of decoding the multiuser estimators do not have knowledge of the signals received by the other multiuser estimators or the information derived by the FEC decoders. Performance greatly improves during the second iteration because now each MUD has a priori information derived from the other MUDs and FEC decoders. Performance continues to improve as additional iterations of turbo processing are executed, but the incremental gains quickly diminish during subsequent iterations.

Figure 4 shows that the performance of the conventional matched filter receiver is worse for low C/I, but improves as the interference diminishes. The performance of macrodiversity alone (without MUD) is relatively invariant to changes in C/I. However, when macrodiversity is used in conjunction with combined MUD/FEC-decoding performance actually improves as interference increases and the C/I ratio gets smaller. This is because the system considers all interference to be useful information, and can exploit high levels of interference to improve overall system performance.

V. Conclusion and Future Work

The proposed turbo-processing based method for combining macrodiversity, MUD, and FEC-decoding shows a significant improvement over the conventional receiver, both with and without macrodiversity combining. By performing just two iterations of decoding and only considering the first tier of cochannel interferers, the performance of a TDMA cellular system can be greatly improved. Performance actually improves as the interference increases. This behavior suggests a possible reduction in the frequency reuse distance. Also because macrodiversity is used, performance improves at cell boundaries, which in turn lowers the outage probability.

In this paper it is assumed that perfect estimates are available for the channels between each transmitter and each receiver. While it is possible to obtain reliable estimates of the channel of the strongest user in each cell, it is much more difficult to obtain estimates of the interferers' channels. Thus, the issue of channel estimation in the context of the proposed system should be more closely investigated. Additionally, the simulations only considered the effect of the first tier of interferers. Since the proposed technique is able to greatly reduce the effect of first tier interference, the effect of interference from outer tiers needs to be studied. Finally the impact on network architecture (cell layout, handoff, etc.) and outage probability should be considered.

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